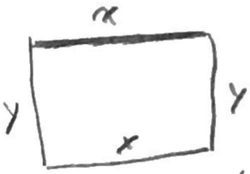
① Cost = (cost of fencing along road)
+ (cost of fencing on other 3 sides)
= (length of fence along road) × (cost per foot)
+ (length of other 3 sides) × (cost per foot)

$C = x \cdot 3 + (x+y+y) \cdot 1 = 4x + 2y$

A = area of pen = xy



Let x and y be the dimensions of the rectangle. The side along the road has length x

a) Maximize $A = xy$ with constraint Cost = 200. So, $200 = 4x + 2y$. Also, $x \geq 0$ and $y \geq 0$
Solve the constraint for y: $2y = 200 - 4x$, $y = 100 - 2x$.

So $A = xy = x(100-2x) = 100x - 2x^2$, $0 \leq x \leq 50$
[$y \geq 0 \Rightarrow 100 - 2x \geq 0$ so $x \leq 50$]

$\frac{dA}{dx} = 100 - 4x$

Critical point occurs when $100 - 4x = 0$, or $x = 25$.
The area at $x = 25$ is $100 \cdot 25 - 2 \cdot 25^2 = 2500 - 2 \cdot 625$
$= 1250$

At the endpoints, ($x = 0$ and $x = 50$), A is 0.
So, $\underline{1250\ ft^2 \text{ is the maximum area}}$. [It occurs when $x = 25$ ft and $y = 50$ ft, but the problem does not ask for the dimensions.]

b) Minimize cost subject to the constraint that area = 800
Minimize $C = 4x + 2y$ where $800 = xy \Rightarrow y = \frac{800}{x}$.

$C = 4x + 2 \cdot \frac{800}{x} = 4x + \frac{1600}{x}$, $x > 0$

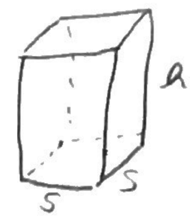
$\frac{dC}{dx} = 4 - \frac{1600}{x^2}$, Critical point: $4 - \frac{1600}{x^2} = 0 \Rightarrow 4 = \frac{1600}{x^2}$
$\Rightarrow 4x^2 = 1600 \Rightarrow x^2 = 400 \Rightarrow x = 20$. When $x = 20$,
$C = 4 \cdot 20 + \frac{1600}{20} = 80 + 80 = 160$. $\frac{d^2C}{dx^2} = \frac{3200}{x^3} > 0$ when $x > 0$,
so $\underline{C = \$160}$ is a minimum by 2nd derivative test.

(2) Minimize $A = 2s^2 + 4sh$, with the constraint $V = s^2 h = 8$. Solving the constraint for h gives $h = \frac{8}{s^2}$, and

$A = 2s^2 + 4s \cdot \left(\frac{8}{s^2}\right) = 2s^2 + \frac{32}{s}$, $s > 0$.

$\frac{dA}{ds} = 4s - \frac{32}{s^2}$. The critical point occurs when $4s - \frac{32}{s^2} = 0$, or $4s^3 = 32$, or $s^3 = 8$, or $s = 2$. When $s = 2$,

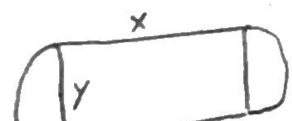
$h = \frac{8}{s^2} = \frac{8}{4} = 2$ also. $\frac{d^2A}{ds^2} = \frac{64}{s^3} > 0$ on the domain $s > 0$. So the function is always concave up, and $s = 2$ ft, $h = 2$ ft gives a minimum volume.



Let s be the length of the side of the square bottom and let h be the height. Then

Area $= s^2 + s^2 + sh + sh + sh + sh$
$= 2s^2 + 4sh$

(3) Maximize $A = xy$ subject to the constraint that the perimeter is 440 meters. The perimeter consists of two sides of length x plus the circumference of a circle of diameter y, so the constraint is $440 = 2x + \pi y$.



Let y be the side of the rectangle that is also a diameter of the semicircle, and let x be the other side of the rectangle.

Solving for x, $x = \frac{440 - \pi y}{2}$,

So $A = xy = \left(\frac{440 - \pi y}{2}\right) y = \frac{1}{2}(440y - \pi y^2)$,

y must be ≥ 0, and we must also have $\pi y \leq 440$,

So, maximize $A = \frac{1}{2}(440y - \pi y^2)$ on the interval $0 \leq y \leq \frac{440}{\pi}$.

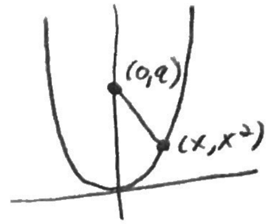
$\frac{dA}{dy} = \frac{1}{2}(440 - 2\pi y)$. Critical point occurs when

$440 - 2\pi y = 0$, or $y = \frac{220}{\pi}$. At that point,

$A = \frac{1}{2}\left(440 \cdot \frac{220}{\pi} - \pi\left(\frac{220}{\pi}\right)^2\right) = \frac{1}{2\pi} \cdot 220(440 - 220) = \frac{24200}{\pi} \approx 7703.1$.

At the two end points, $A = 0$, so the maximum value is $A = \frac{24200}{\pi}$ square meters ≈ 7703.1 square meters.

(4) We want to minimize distance $D = \sqrt{(x-0)^2 + (x^2-a)^2}$. It suffices to minimize the square of the distance,



$$S = x^2 + (x^2-a)^2$$
$$= x^2 + x^4 - 2ax^2 + a^2$$
$$= x^4 + (1-2a)x^2 + a^2 \quad \text{[with no restriction on } x\text{]}$$

$$\frac{dS}{dx} = 4x^3 + 2(1-2a)x = 2x\,(2x^2 + (1-2a))$$

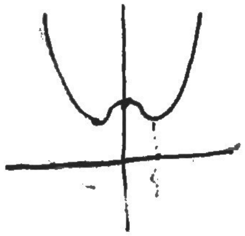
The critical points occur when $2x = 0$ or $2x^2 + (1-2a) = 0$. The second equation says $2x^2 = -(1-2a) = 2a-1$, which only has a solution when $2a-1 \geq 0$. In that case, $x^2 = \frac{2a-1}{2}$, $x = \pm\sqrt{\frac{2a-1}{2}}$

Case $2a-1 \geq 0$
$a \geq \frac{1}{2}$: The critical points are $0, \sqrt{\frac{2a-1}{2}}, -\sqrt{\frac{2a-1}{2}}$

Using the first derivative test:

x		$-\sqrt{\frac{2a-1}{2}}$		0		$\sqrt{\frac{2a-1}{2}}$	
$f'(x)$	$---$	0	$++++$	0	$---$	0	$++++$
f:	$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$

(The graph of S looks like



we see $x=0$ gives a local maximum, and $x = \pm\sqrt{\frac{2a-1}{2}}$ gives a minimum. The value at both points is $\sqrt{\frac{2a-1}{2}}$

Case $2a-1 < 0$
$a \leq \frac{1}{2}$: The only critical point is $x = 0$.
$\frac{d^2S}{dS^2} = 12x^2 + 2(1-2a)$, which is always positive because $1-2a > 0$.
So $x=0$ gives a minimum because the graph is always concave up.

(5) Maximize xy^2, subject to $x+y=1$, $x \geq 0$, $y \geq 0$.

Since $x+y=1$, $x = 1-y$, and we want to maximize $z = (1-y)y^2 = y^2 - y^3$, where $0 \leq y \leq 1$.

$\frac{dz}{dy} = 2y - 3y^2$, so the critical point occurs when $y(2-3y) = 0$, that is, when $y = 0$ or $y = \frac{2}{3}$

Checking the endpoints and the critical points, $z = 0$ when $y = 0$ or $y = 1$, and $z > 0$ when $y = \frac{2}{3}$.

So the maximum occurs when $y = \frac{2}{3}$ and $x = \frac{1}{3}$.

(6) Maximize $z = x^n y^m$ subject to $x+y=1$, $x \geq 0$, $y \geq 0$, and $0 \leq x \leq 1$.

Then $z = x^n (1-x)^m$,

$$\frac{dz}{dx} = x^n \frac{d}{dx}(1-x)^m + (1-x)^m \frac{d}{dx} x^n$$

$$= x^n \cdot m(1-x)^{m-1} \cdot \frac{d}{dx}(1-x) + (1-x)^m \cdot n x^{n-1}$$

$$= m x^n (1-x)^{m-1}(-1) + n(1-x)^m \cdot x^{n-1}$$

$$= -m x \cdot x^{n-1}(1-x)^{m-1} + n(1-x)(1-x)^{m-1} x^{n-1}$$

$$= x^{n-1}(1-x)^{m-1}[-mx + n(1-x)]$$

$$= x^{n-1}(1-x)^{m-1}[-mx + n - nx]$$

$$= x^{n-1}(1-x)^{m-1}[n - (n+m)x]$$

Critical points occur when $x=0$, $x=1$, or $n-(n+m)x=0$. That is $x=0$, $x=1$, $x = \frac{n}{n+m}$. $z=0$ at $x=0$ and $x=1$ and is positive when $x = \frac{n}{n+m}$, so the maximum occurs when $x = \frac{n}{n+m}$ and $y = 1 - \frac{n}{n+m} = \frac{(n+m)-n}{n+m} = \frac{m}{n+m}$.