Math 130-01, Spring 2020 Sample Answers to Homework # 3

① Suppose that $\lim_{x\to a} f(x) = M$, and c is a constant. We want to show that $\lim_{x\to a} cf(x)$ exists and is equal to cM.

Let $\varepsilon > 0$. We need to find a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|(c\,f(x)) - (cM)| < \varepsilon$. Since $\lim_{x\to a} f(x) = M$, we can find a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - M| < \varepsilon/|c|$

For this same δ, if $0 < |x-a| < \delta$, then

$$|(cf(x)) - (cM)| = |c\cdot(f(x) - M)| = |c|\cdot|f(x) - M| < |c|\cdot \varepsilon/|c| = \varepsilon.$$

② a) average velocity $= \dfrac{\text{change in distance}}{\text{change in time}} = \dfrac{60 \text{ miles}}{1 \text{ hour}} = 60 \text{ miles/hour}$.

b) If my velocity was less than 60 miles/hour at all times, then I could not have covered a full 60 miles in one hour.

c) Physically, velocity can't jump from one value to another — it would have to pass through all the velocities in between, and that makes the velocity function continuous.

d) At the start, my velocity is zero. Part b shows that at some time, t_0, my velocity is ≥ 60. If velocity at t_0 is 60, we've found the time we want. If the velocity at t_0 is greater than 60, then velocity is a continuous function that is less than 60 at $t=0$, greater than 60 at $t=t_0$. By the IVT, there is a time in the interval $[0,t_0]$ at which the velocity equals 60.

③ a) $$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3} \frac{(2x-1) - (2\cdot 3 - 1)}{x-3}$$

$$= \lim_{x\to 3} \frac{2x-6}{x-3} = \lim_{x\to 3} \frac{2(x-3)}{x-3} = \underline{\underline{2}}$$

b) $f'(-2) = \lim_{x \to -2} \dfrac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2} \dfrac{3x^2 - 3(-2)^2}{x+2}$

$= \lim_{x \to -2} \dfrac{3(x^2-4)}{x+2} = \lim_{x \to -2} \dfrac{3(x-2)(x+2)}{x+2} = \lim_{x \to -2} 3(x-2) = \underline{\underline{-12}}$

c) $f'(1) = \lim_{x \to 1} \dfrac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \dfrac{\frac{1}{x+1} - \frac{1}{1+1}}{x-1} = \lim_{x \to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x+1} - \dfrac{1}{2}\right)$

$= \lim_{x \to 1} \dfrac{1}{x-1}\left(\dfrac{2-(x+1)}{2(x+1)}\right) = \lim_{x \to 1} \dfrac{1}{x-1}\left(\dfrac{-x+1}{2(x+1)}\right)$

$= \lim_{x \to 1} \dfrac{1}{x-1}\left(\dfrac{-(x-1)}{2(x+1)}\right) = \lim_{x \to 1} \dfrac{-1}{2(x+1)} = \underline{\underline{-\dfrac{1}{4}}}$

4 a) $f'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{(2(2+h) - 3(2+h)^2) - (2 \cdot 2 - 3 \cdot 2^2)}{h}$

$= \lim_{h \to 0} \dfrac{4 + 2h - 3(4+4h+h^2) - (-8)}{h}$

$= \lim_{h \to 0} \dfrac{4+2h-12-12h-3h^2+8}{h} = \lim_{h \to 0} \dfrac{-10h-3h^2}{h} = \lim_{h \to 0} -10-3h = \underline{\underline{-10}}$

b) $f'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \dfrac{(1+h)^3 - 1^3}{h}$

$= \lim_{h \to 0} \dfrac{1+3h+3h^2+h^3-1}{h} = \lim_{h \to 0} 3+3h+h^2 = \underline{\underline{3}}$

c) $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{\sqrt{h+9} - \sqrt{9}}{h}$

$= \lim_{h \to 0} \left(\dfrac{\sqrt{h+9} - 3}{h} \cdot \dfrac{\sqrt{h+9}+3}{\sqrt{h+9}+3}\right) = \lim_{h \to 0} \dfrac{(h+9) - 9}{h(\sqrt{h+9}+3}$

$= \lim_{h \to 0} \dfrac{1}{\sqrt{h+9}+3} = \dfrac{1}{\sqrt{9}+3} = \dfrac{1}{6}$