Math 130, Spring 2020 Sample Answers to Homework #8

① a) $\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}x^3$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 3x^2\frac{dx}{dt}$

b) $\frac{d}{dt}(xe^y + ye^x) = \frac{d}{dt}(10)$

$\left(x\frac{d}{dt}e^y + e^y\frac{d}{dt}x\right) + \left(y\frac{d}{dt}e^x + e^x\frac{d}{dt}y\right) = 0$

$xe^y\frac{dy}{dt} + e^y\frac{dx}{dt} + ye^x\frac{dx}{dt} + e^x\frac{dy}{dt} = 0$

c) $\frac{d}{dt}(\sin(xy)) = \frac{d}{dt}\left(\frac{1}{2}\right)$

$\cos(xy)\frac{d}{dt}(xy) = 0$

$\cos(xy)\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = 0$

② $\frac{d}{dt}(x^2y^3 + y) = \frac{d}{dt}(40)$

$\left(x^2\frac{d}{dt}y^3 + y^3\frac{d}{dt}x^2\right) + \frac{d}{dt}y = 0$

$x^2\cdot 3y^2\frac{dy}{dt} + y^3\cdot 2x\frac{dx}{dt} + \frac{dy}{dt} = 0$

$\underline{(3x^2y^2+1)\frac{dy}{dt} + 2xy^3\frac{dx}{dt} = 0}$

At the time in question:

$y=2$, so $x^2\cdot 2^3 + 2 = 40$

$8x^2 = 38$

$x^2 = \frac{38}{8} = \frac{19}{4}$

$\underline{x = \pm\sqrt{\frac{19}{4}} = \pm\frac{\sqrt{19}}{2}}$

$\frac{dy}{dt} = 4$

$\left(3\cdot\left(\pm\frac{\sqrt{19}}{2}\right)^2 2^2 + 1\right)\cdot 4 \pm 2\frac{\sqrt{19}}{2}\cdot 2^3\frac{dx}{dt} = 0$

$\frac{dx}{dt} = \pm\frac{(3\cdot\frac{19}{4}\cdot 4+1)4}{8\sqrt{19}}$

$= \pm\frac{58}{2\sqrt{19}} = \pm\frac{29}{\sqrt{19}}$

$\approx \underline{6.653}$

(3)

$V = \frac{4}{3}\pi r^3$

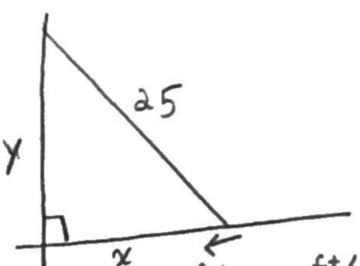
$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

when $r = 20$ inches and $\frac{dr}{dt} = 2$ inches per second,

$$\frac{dV}{dt} = 4\pi \cdot (20)^2 \cdot 2 = \underline{3200\pi \text{ in}^3/\text{sec}}$$

$$\approx \underline{10{,}053.1 \text{ in}^3/\text{sec}}$$

(4)



$\frac{dx}{dt} = -1$ ft/sec

Let x be the distance of the bottom of the ladder from the wall. Let y be the height of the top of the ladder above the ground.

$x = 20$ ft at $t = 0$

At $t = 5$, $x = \underline{15}$ ft

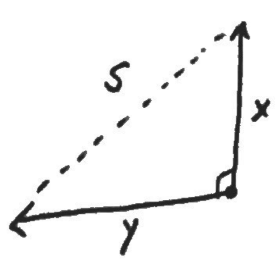
$$y = \sqrt{25^2 - 15^2}$$
$$= \sqrt{625 - 225}$$
$$= \sqrt{400}$$
$$= 20\text{ft}$$

$$x^2 + y^2 = 25$$
$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}25$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

At time $t = 5$, $x = 15, y = 20, \frac{dx}{dt} = -1$

and $\frac{dy}{dt} = -\frac{15}{20}(-1) = \underline{\frac{3}{4}\text{ ft/sec}}$

(5)



Let x be the distance walked by the first person, y the distance walked by the second person. Let s be the distance between the two people. Then

$\frac{dx}{dt} = 3$ mi/hr

$\frac{dy}{dt} = 4$ mi/hr

At $t = 2$,

$x = 6$ miles

$y = 8$ miles

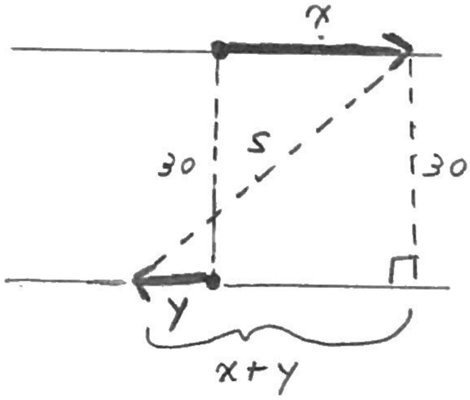
$s = \sqrt{6^2 + 8^2} = 10$mi

$$s^2 = x^2 + y^2$$
$$\frac{d}{dt}s^2 = \frac{d}{dt}(x^2 + y^2)$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

At $t = 2$:

$$10\frac{ds}{dt} = \frac{6 \cdot 3 + 8 \cdot 4}{10} = \frac{50}{10} = \underline{5 \text{ mi/hr}}$$

⑥



Let x be the distance walked by Wilma, y the distance walked by Fred. Let s be the distance between them. Then, as shown in the diagram,

$$(x+y)^2 + 30^2 = s^2$$

$$\frac{d}{dt}(x+y)^2 + \frac{d}{dt}30^2 = \frac{d}{dt}s^2$$

$$2(x+y)\frac{d}{dt}(x+y) + 0 = 2s\frac{ds}{dt}$$

$$(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = s\frac{ds}{dt}$$

$$\frac{dx}{dt} = 7, \quad \frac{dy}{dt} = 3$$

At $t = 2$, $x = 14$, $y = 6$

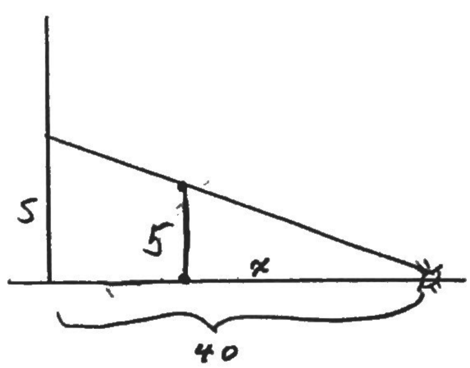
$$s = \sqrt{30^2 + (14+6)^2}$$
$$= \sqrt{900 + 400}$$
$$= \sqrt{1300}$$
$$= 10\sqrt{13}$$

At $t = 0$:

$$(14+6)(7+3) = 10\sqrt{13}\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{200}{10\sqrt{13}} = \frac{20}{\sqrt{13}} \text{ ft/sec} \approx 5.547 \frac{\text{ft}}{\text{sec}}$$

⑦



Let x be the distance of the person from the spotlight (so $40 - x$ is the distance of the person from the wall). Let s be the height of the shadow on the wall. Then by similar triangles,

$$\frac{s}{40} = \frac{5}{x}$$

$$s = \frac{200}{x}$$

$$\frac{ds}{dt} = -\frac{200}{x^2}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2 \text{ ft/sec}$$

when the person is 10 ft from the wall, $x = 30$

and $\frac{s}{40} = \frac{5}{30}$ so

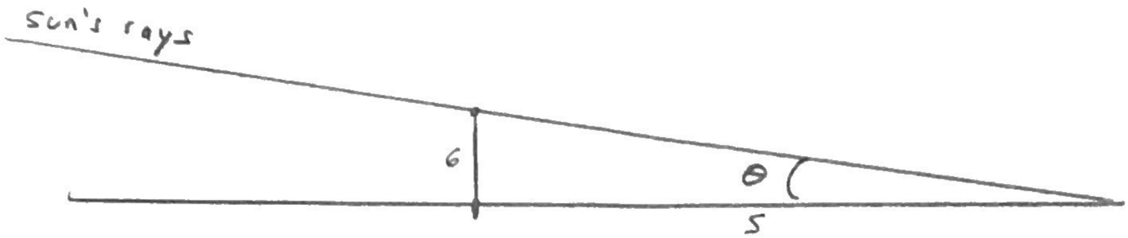
$$s = \frac{40 \cdot 5}{30} = \frac{20}{3} \text{ ft}$$

[but we don't need s in the end]

when $x = 30$:

$$\frac{ds}{dt} = -\frac{200}{30^2} \cdot 2 = -\frac{4}{9} \text{ ft/sec}$$

(5)



Let s be the length of the shadow.
Let θ be the angle between the rays from the sun and the plane. Then

Alternatively:

$\cot\theta = \frac{s}{6}$

$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{6}\frac{ds}{dt}$

$\frac{ds}{dt} = -6\csc\theta \frac{d\theta}{dt}$

$$\tan\theta = \frac{6}{s}$$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{6}{s}$$

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$$

So $$\frac{ds}{dt} = -\frac{s^2}{6}\sec^2\theta \frac{d\theta}{dt}$$

At the time in question,

$$\theta = 5° = \frac{5}{180}\pi$$

$$s = \frac{6}{\tan\theta} = \frac{6}{\tan\left(\frac{5}{180}\pi\right)}$$

$$\frac{ds}{dt} = -\frac{6^2\sec^2\left(\frac{5}{180}\pi\right)}{6\cdot\tan^2\left(\frac{5}{180}\pi\right)}\cdot\frac{d\theta}{dt}$$

$$= -6\cdot\csc^2\left(\frac{5}{180}\pi\right)\frac{d\theta}{dt}$$

What I expect you to say:

Since the sun goes around the Earth once every day, $\frac{d\theta}{dt} = 2\pi \text{ radians}/24 \text{ hours}$

$$= \frac{2\pi}{24\cdot60\cdot60} \text{ radians/sec}$$

$$\approx 0.00007272 \text{ radians/sec}$$

And when $\theta = \frac{5\pi}{180}$,

$$\frac{ds}{dt} = -6\cdot\csc^2\left(\frac{5\pi}{180}\right)\cdot\frac{2\pi}{86400} \text{ ft/sec}$$

$$\approx 0.0574 \text{ ft/sec} = \underline{0.693 \text{ in/sec}}$$

(But in fact, this is only true if the sun is setting perpendicular to the horizon, for example if Barney is on the equator on an equinox. In general, you would need to take into account the latitude and time of year.)