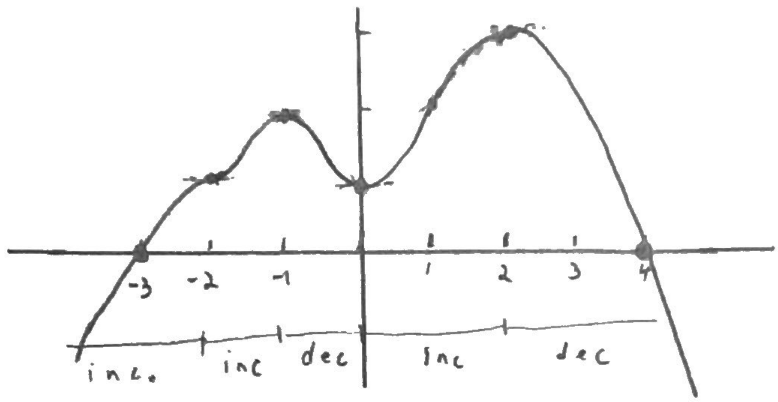
Math 130, Spring 2020 Sample Answers to Homework 9

① $p(x) = ax^2+bx+c$, $p'(x) = 2ax+b$. The critical point is found by solving $p'(x)=0$, or $2ax+b=0$, so $x = -\frac{b}{2a}$ is the critical point. $p''(x) = 2a$. If $a>0$, $p(x)$ is always concave up, and the critical point gives a minimum, If $a<0$, $p(x)$ is always concave down and the critical point gives a local minimum.

②



③ $g(x) = x - \sin(x)$, $g'(x) = 1-\cos(x)$. Since $-\cos(x) \geq -1$ for all x, $1-\cos(x) \geq 0$ for all x. So $g(x)$ is increasing for all x except for its critical points, where $g'(x)=0$.
($g'(x)=0$ means $1-\cos(x)=0$, or $\cos(x)=-1$.
$\cos(x) = -1$ when $x=0$ but also when x differs from 0 by a multiple of 2π. So the critical points are
$x = 2k\pi$ for all integers k.)

④ $f(x) = 2x^3+9x^2+12$

$f'(x) = 6x^2+18x = 6x(x+3)$,
This is 0 when $x=0$, $x=-3$

x: −3 0
$f'(x)$: +++ 0 − − − − − 0 + + + +

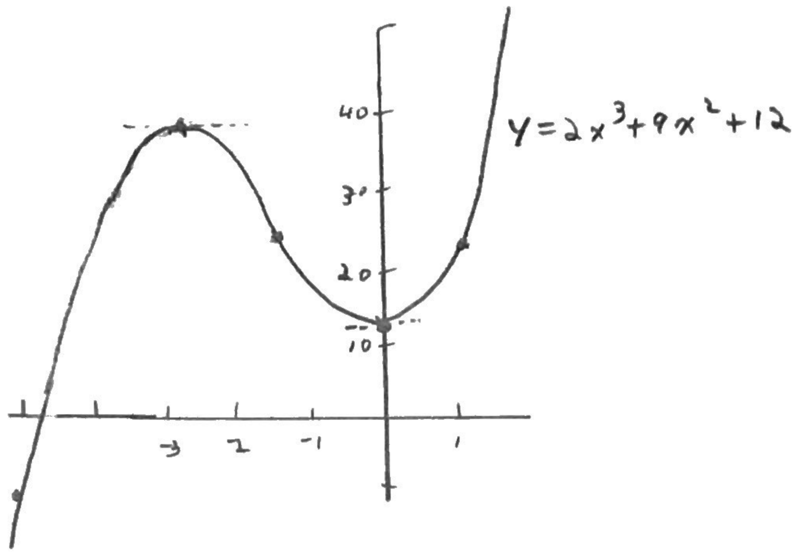
increasing for $x<-3$, $x>0$
decreasing for $-3<x<0$
local max at $x=-3$
local min at $x=0$

$f''(x) = 12x+18 = 6(2x+3)$
$f''(x)=0$ when $x = -\frac{3}{2}$

x: −3/2
$f''(x)$: − − − 0 + + + +

$f(x)$ is concave down for $x<-\frac{3}{2}$
concave up for $x>-\frac{3}{2}$
inflection point at $x=-\frac{3}{2}$

x	$f(x)$	$f'(x)$	$f''(x)$
-3	39		
0	12	0	18
$-3/2$	$\frac{51}{2}$		0
1	23	24	30
-4	28		
-5	-13		



⑤ $p(x) = x^4 - 2x^2 = x^2(x^2-2)$

$p'(x) = 4x^3 - 4x = 4x(x^2-1)$

$p'(x) = 0$ if $x = 0, 1, -1$

$$\begin{array}{lccccccc} x & & -1 & & 0 & & 1 & \\ p'(x) & --- & 0 & +++ & 0 & --- & 0 & ++++ \end{array}$$

$f(x)$ is increasing for $-1 < x < 0$, $x > 1$

decreasing for $x < -1$, $0 < x < 1$

local mins at $x = -1, 1$

local max at $x = 0$

$p''(x) = 12x^2 - 4 = 4(3x^2 - 1)$

$p''(x) = 0$ if $x = \pm\sqrt{\frac{1}{3}}$

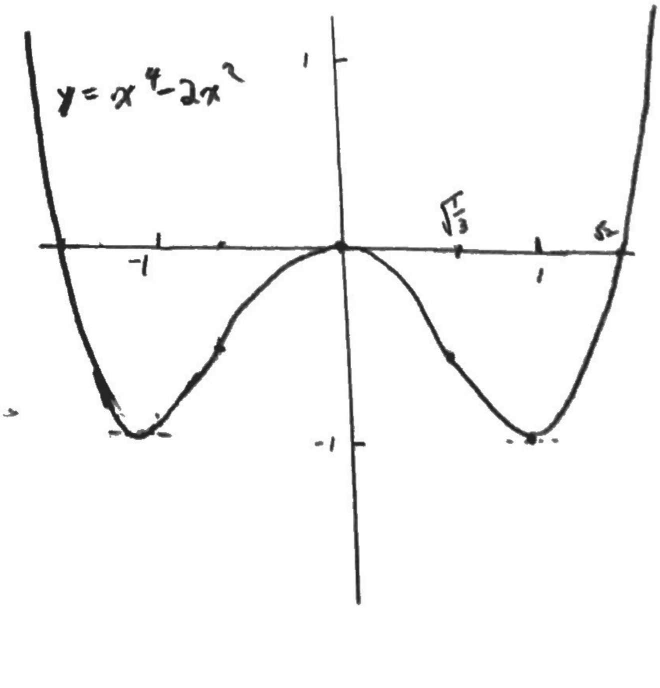
$$\begin{array}{lccccc} x & & -\sqrt{\frac{1}{3}} & & +\sqrt{1/3} & \\ p''(x) & +++ & 0 & -------- & 0 & ++++ \end{array}$$

$p(x)$ is concave up for $x < -\frac{1}{\sqrt{3}}$,
$x > \frac{1}{\sqrt{3}}$

concave down for $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

inflection points at $x = \pm\frac{1}{\sqrt{3}}$

x	$p(x)$	$p'(x)$	$p''(x)$
-1	-1	0	
$-\sqrt{\frac{1}{3}}$	$-\frac{5}{9}$	≈ 1.5	0
0	0	0	
$\sqrt{1/3}$	$-5/9$	≈ -1.5	0
1	-1	0	
2	8	24	
$\sqrt{2}$	0	$4\sqrt{2}$	



⑥ $f(x) = xe^{-x}$

$f'(x) = xe^{-x}(-1) + e^{-x}$

$= (1-x)e^{-x}$

$f'(x) = 0$ if $x = 1$

x		0		1		2	
$f'(x)$	$+ + + + +$			0	$- - - -$		

$f(x)$ increasing for $x < 0$

decreasing for $x > 0$

local max at $x = 1$

$f''(x) = (1-x)e^{-x}(-1) + (-1)e^{-x}$

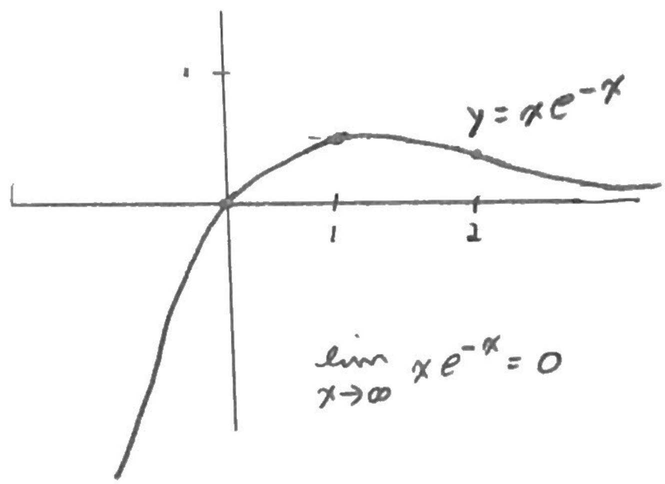
$= (x-2)e^{-x}$

$f(x)$ concave down for $x < 2$

concave up for $x > 2$

inflection point at $x = 2$

x	$f(x)$	$f'(x)$	$f''(x)$
0	0	$1/e$	$-2/e$
1	$1/e$	0	$-1/e$
2	$2/e^2$	$-1/e^2$	0
-1	$-e$	$2e$	



$\lim_{x \to \infty} xe^{-x} = 0$

⑦ $f(x) = \frac{x}{1+x^2}$

$f'(x) = \frac{(1+x^2)\cdot 1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

$f'(x) = 0$ if $x = \pm 1$

x		-1		1	
$f'(x)$	$- - - -$	0	$+ + + +$	0	$- - - - -$

increasing for $-1 < x < 1$

decreasing for $x < -1$, $x > 1$

local min at $x = -1$

local max at $x = 1$

$f''(x) = \frac{(1+x^2)^2 \cdot (-2x) - (1-x^2)\,2(1+x^2)(2x)}{(1+x^2)^4}$

$= \cdots = \frac{2x(x^2-3)}{(1+x^2)^3} = 0$ if $x = 0, \pm\sqrt{3}$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$f''(x)$	$- - -$	0	$+ + + +$	0	$- - - -$	0	$+ + + + +$

$f(x)$ is concave up for $-\sqrt{3} < x < 0$, $x > \sqrt{3}$

concave down for $-\sqrt{3} < x < \sqrt{3}$

inflection points at $\pm\sqrt{3}$

x	$f(x)$	$f'(x)$	$f''(x)$
$-\sqrt{3}$	$-\sqrt{3}/4$		0
-1	$-1/2$	0	
0	0	1	0
1	$1/2$	0	
$\sqrt{3}$	$\sqrt{3}/4$		0