This lab is due at the start of next week's lab.

1. The idea of a *differential equation* was just introduced in class yesterday, and we looked at the idea of a *separable first-order differential equation*. Such equations can be solved by separating the variables and then integrating both sides of the equation. Use separation of variables to find all solutions to the following differential equations. Your answers should be in the form $y = f(t)$, where the function $f(t)$ will involve some arbitrary constant.

a) $\dfrac{dy}{dt} = \dfrac{t^2}{y}$ **b)** $2ty\dfrac{dy}{dt} = 1$

2. The solution to a differential equation can include one or more arbitrary constants. Sometimes, we are given more information, such as the value of the function at a certain point. A differential equation with this extra information is called an *initial value problem*. Solve the initial value problem $2ty\dfrac{dy}{dt} = 1$, where $y(1) = 3$. That is, find the value of the arbitrary constant in the solution that will make $y(1) = 3$. Note that you have already found the general solution of the differential equation in the previous problem, so this problem is **very** easy. It's here mainly to introduce the idea of an initial value problem.

3. Consider the improper integral $\int_0^\infty x^n e^{-x}\,dx$, where n is a non-negative integer. For the case $n = 0$, we showed in class that $\int_0^\infty e^{-x}\,dx = 1$. For $n > 0$, we can apply the reduction formula $\int x^n e^{-1} dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}dx$ to deduce that for any $b > 0$,

$$\int_0^b x^n e^{-x}dx = -x^n e^{-x}\Big|_0^b + n\int_0^b x^{n-1}e^{-x}dx$$

Furthermore, we can apply L'Hôpital's Rule multiple times to show that

$$\lim_{b\to\infty} b^n e^{-b} = \lim_{b\to\infty}\frac{b^n}{e^b} = \lim_{b\to\infty}\frac{nb^{n-1}}{e^b} = \cdots = \lim_{b\to\infty}\frac{n\cdot(n-1)\cdots 2\cdot b}{e^b} = \lim_{b\to\infty}\frac{n!}{e^b} = 0$$

a) Use these facts to compute $\int_0^\infty xe^{-x}dx$

b) Now, use these facts and the result of part **a)** to compute $\int_0^\infty x^2e^{-x}dx$

c) Now, use these facts and the result of part **b)** to compute $\int_0^\infty x^3e^{-x}dx$

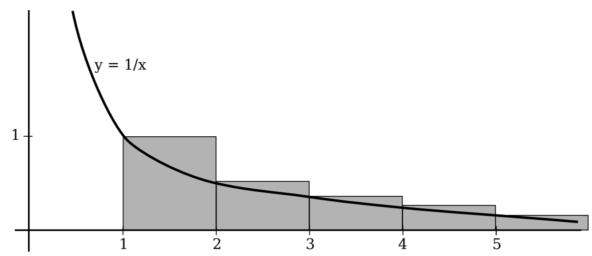
d) What is the value of $\int_0^\infty x^4e^{-x}dx$?

e) What is the value of $\int_0^\infty x^ne^{-x}dx$ for any positive integer n? Why?

(*continued on the other side*)

4. We will soon be looking at *infinite series*, that is, sums such as $\sum_{k=1}^{\infty} a_n$. Infinite series have a lot in common with improper integrals of the form $\int_a^{\infty} f(x)\,dx$. Consider for example, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$.

a) For a positive integer N, consider the left Riemann sum for $f(x) = 1/x$ on the interval $[1, N+1]$, using N subintervals (so that $\Delta x = 1$):



Explain why this shows that $\sum_{k=1}^{N} \frac{1}{k} \;>\; \int_1^{N+1} \frac{1}{x}\,dx$

b) Use a right Riemann sum on the interval $[1, N]$ to show $\int_1^{N} \frac{1}{x}\,dx \;>\; \sum_{k=2}^{N} \frac{1}{k}$

(Note the "$k = 2$" in the sum! Draw a picture!)

c) Deduce that $\ln(N+1) \;<\; \sum_{k=1}^{N} \frac{1}{k} \;<\; 1 + \ln(N)$

d) Is the sum $\sum_{k=1}^{\infty} \frac{1}{k}$ finite or is it infinite? Why? (You only need one of the inequalities from part **c)** to answer this question.)

e) Do a similar analysis for $\sum_{k=1}^{\infty} \frac{1}{k^2}$, comparing it to $\int_1^{\infty} \frac{1}{x^2}\,dx$