This lab is due at the start of next week's lab.

1. Determine whether each of the following sequences converges or diverges. If it converges, find the limit of the sequence as $n \to \infty$.

a) $\left\{ \frac{3^n}{7^n} \right\}_{n=1}^{\infty}$ **b)** $\left\{ (-1)^i \frac{i}{i-3} \right\}_{i=4}^{\infty}$ **c)** $\left\{ \frac{2k+1}{1000000} \right\}_{k=1}^{\infty}$ **d)** $\left\{ 2 + \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$

2. For any given number x, the infinite series $\sum_{n=0}^{\infty} \frac{7x^n}{3^n}$ is a geometric series. What is the initial term, and what is the ratio? For which values of x does the series converge? When it converges, what does it converge to? (The answer to the last question is a function of x. This shows that some functions of x can be written as infinite series involving powers of x. Such series are called *power series.*)

3. In the previous lab, you showed that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln(n) + 1$$

Use this fact to estimate $\sum_{i=1}^{1000} \frac{1}{i}$ and $\sum_{i=1}^{1000000} \frac{1}{i}$, as closely as possible. Obtain both an upper bound and a lower bound for the sum, and compute the difference between the two bounds. (Use a calculator for this one, to get actual numbers.) About how large will N have to be in order to get have $\sum_{i=1}^{N} \frac{1}{i} > 100$? Explain your reasoning!

4. One of Zeno's famous paradoxes goes something like this: Achilles is in a race with a tortoise. Achilles runs at a speed of 10 meters per second, while the tortoise plods along at just 1 meter per second. The tortoise is given a 100 meter head start at the beginning of the race. Zeno argues that Achilles must lose the race because he can never even catch up with the tortoise:

- At 10 seconds, Achilles reaches the tortoise's original position, but by that time, the tortoise has moved forward another 10 meters, so is still in the lead.
- After another 1 second, Achilles has covered the ten meters, but the tortoise has moved forward another meter, so is still in the lead.
- After another 1/10 second, Achilles has covered the meter, but the tortoise has moved forward another 1/10 meter, so is still in the lead.

Clearly, this continues forever. Achilles reaches a point where the tortoise *used to be*, but the tortoise has already moved ahead some additional distance and so is still in the lead! This happens over and over, infinitely, so Achilles never overtakes the tortoise.

Explain (**in words!**) how this paradox can be resolved by using an infinite series. Write down a series that sums up the sequence of distances traveled by Achilles, and use the series to find the exact distance where Achilles catches up to the tortoise.

5. For this problem, suppose that when a certain ball falls from a given height, it bounces back up to exactly half of its previous height. (This is all done in an airless chamber so that there is no air resistance.) Suppose that the ball is dropped from a height of 8 feet. Then on the first bounce, it rises to a height of 4 feet, on the second to a height of 2 feet, and so on. According to this model, the ball will bounce an infinite number of times. For all parts of this problem, **you must explain your work**.

a) Find the total distance traveled by the ball, starting from the time that it first hits the ground by finding the sum of an infinite series. (To make things a little simpler, we ignore the initial fall and just start computing from the time it first hits the ground.)

b) When an object is falling freely near the surface of the Earth (ignoring air resistance), its height is given by a function of the form $y = -16t^2 + v_o t + y_o$, where time is measured in seconds, height is measured in feet, v_o is the velocity at time zero, and y_o is the height at time zero. This equations holds for our ball **between bounces**. For a bounce, the initial height is zero, so the formula is $y = -16t^2 + v_o t$. Suppose that a ball bounces to a height of h. What was its initial velocity, v_o, as it left the ground? The answer will be in terms of h. (Start by finding the time, t, when the ball reaches its greatest height. Measure time from the moment when the ball leaves the ground on the way up.) This allows you to write v_o in terms of h in the equation $y = -16t^2 + v_o t$.

c) Still assuming that the ball bounces to a height of h, how long does the bounce take? The answer will be in terms of h. Use the equation for y from the previous part, with v_o written in terms of h. (You will need to solve $y(t) = 0$, since that will give you the time when the ball strikes the ground.)

d) Returning to the bouncing ball, write down an infinite series for the **total time** for which the ball bounces (from the time it first hits the ground). Find the sum of the series, or show that it diverges to infinity. Does the ball bounce forever?

e) What do you suppose happens with a real ball bouncing in the real world (in a vacuum)?