This lab is due at the start of next week's lab.

We now have several tests for convergence and divergence of infinite series...

Geometric Series Test: A geometric series with ratio r diverges if $|r| \geq 1$ and converges if $|r| < 1$. A convergent geometric series converges to $\frac{a}{1-r}$ where a is the initial term and r is the ratio.

Divergence Test (also called the n^{th} term test): If $\lim\limits_{n\to\infty} a_n \neq 0$ (including the case $\lim\limits_{n\to\infty} a_n$ does not exist) then the infinite series $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges.

Integral Test: Suppose that $f(x)$ is a positive, decreasing function for $x \geq 1$, and suppose $a_n = f(n)$ for $n = 1, 2, 3, \ldots$. If the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. If the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges.

p-series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

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1. Determine whether each of the following infinte series converges or diverges. In each case, state which test, or combination of tests, you are using and state why each test applies. If you can find the value of a convergent series, do so! Show your work!

a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3 + 3n + 1}{5n^3 - n}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}$ **c)** $\displaystyle\sum_{n=0}^{\infty} 1.0001^n$

d) $\displaystyle\sum_{n=0}^{\infty} 0.9999^n$ **e)** $\displaystyle\sum_{i=1}^{\infty} ie^{-i}$ **f)** $\displaystyle\sum_{n=0}^{\infty} \sin(n)$

g) $\displaystyle\sum_{n=0}^{\infty} \sin(n\pi)$ **h)** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5^{n+1}}{7^n}$ **i)** $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k^3}$

2. We have talked a little in class about "power series," which are series that contain powers of a variable such as x. I also mentioned that you can differentiate a convergent power series term-by-term. Here is a power series that converges to $\sin(x)$ for all x. Use it to find a power series for $\cos(x)$, and then verify that when you take the derivative of the power series for $\cos(x)$, you get back the power series for - $\sin(x)$.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

3. We have seen that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞, but it does so very slowly. Are there series that diverge to infinity much more slowly? The integral test shows that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ also diverges to infinity. About how many terms of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ do you have to add up to get a sum that is greater than 100? Use the fact that $\displaystyle\sum_{k=2}^{N} \frac{1}{k\ln(k)} > \int_2^{N+1} \frac{1}{x\ln(x)}\,dx$. It might be interesting to look at how many terms it takes to reach a total of at least 5...

4. (Note: This question is about **sequences**, not **series**.) Let's think about *alternating sequences*, in which every other term is negative. Suppose that a_o, a_1, a_2, $\ldots$, is a sequence of positive numbers ($a_n > 0$). Then $\{(-1)^n a_n\}_{n=0}^{\infty}$ is an alternating sequence. This problem explains why $\{(-1)^n a_n\}_{n=0}^{\infty}$ converges if and only if $\{a_n\}_{n=0}^{\infty}$ converges to zero. Some pictures could be useful in your explanations!

a) Suppose that $\displaystyle\lim_{n\to\infty} a_n = 0$. Show that $\displaystyle\lim_{n\to\infty} (-1)^n a_n = 0$ also. (Hint: Use the Squeeze Theorem.)

b) Now suppose that $\displaystyle\lim_{n\to\infty} (-1)^n a_n$ converges to L. Explain why L must be zero. (Hint: If $L > 0$, can the *negative* terms in the sequence get arbitrary close to L? What about if $L < 0$?)

c) Finally, suppose that $\displaystyle\lim_{n\to\infty} (-1)^n a_n = 0$. Explain why $\displaystyle\lim_{n\to\infty} a_n = 0$ also.

d) Use this result to show that $\{\frac{(-1)^n}{n}\}_{n=0}^{\infty}$ converges to 0 and that $\{(-1)^k \cdot \frac{k^2+3k+1}{5k^2+7}\}_{k=1}^{\infty}$ does not converge.