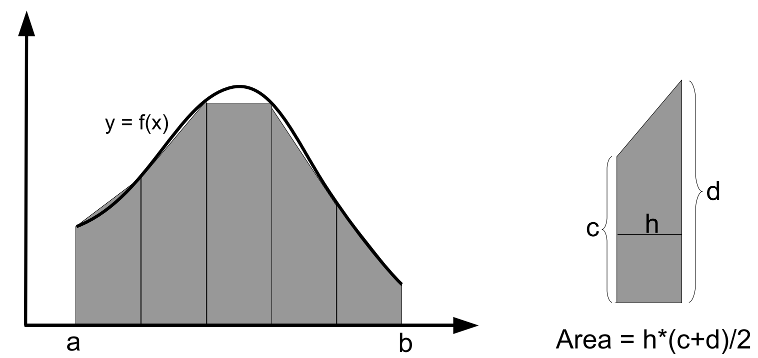
This lab is due at the start of next week's lab.

1. $(x-2)^2+(y-3)^2=2^2$ is a formula for the circle that has radius 2 and center at the point(2,3). Use this fact to find the values of the following definite integrals:

$$\int_0^4 3+\sqrt{4-(x-2)^2}\;dx \qquad\qquad \int_0^4 3-\sqrt{4-(x-2)^2}\;dx$$

Draw pictures! Solve the circle formula for y in terms of x, and use some geometry. Remember that a definite integral represents a certain area!

2. In a Riemann sum, the area under the graph of $y=f(x)$ is approximated by rectangles. A better estimate, using the same number of subintervals, can often be obtained with the *trapezoid rule*, as illustrated below on the left. Let's say that the endpoints of the subintervals are $x_0, x_1, \ldots, x_n$ (where $x_0 = a$ and $x_n = b$). For the trapezoid rule, the area on the i^{th} subinterval, $[x_{i-1}, x_i]$, is approximated by a trapezoid with parallel sides of heights $f(x_{i-1})$ and $f(x_i)$. The upper boundary of the shaded region is generated by drawing lines from one point on the curve to the next.



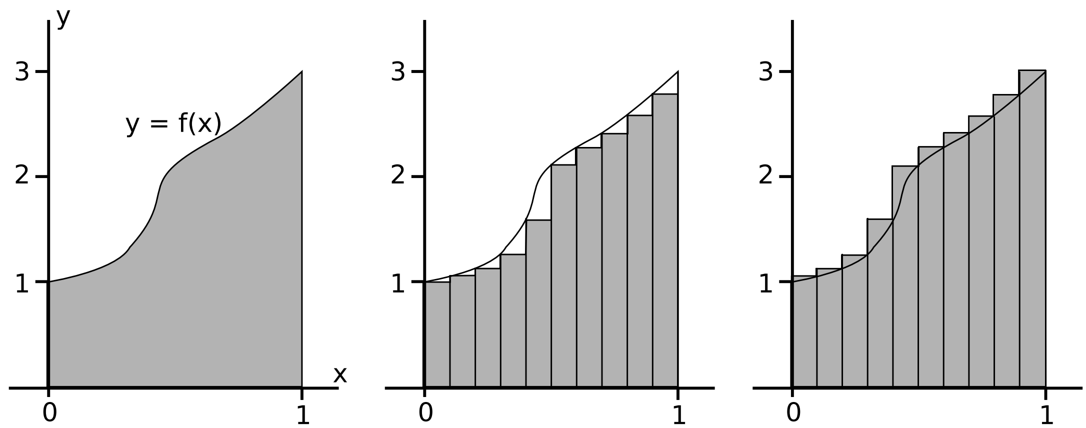
a) Apply the trapezoid rule to estimate the area under the curve $f(x)=4-x^2$ on the interval $[0,2]$, using five subintervals. The formula for the area of a trapezoid is shown in the illustration above on the right.

b) The formula for a left Riemann sum is $f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$ and for the right Riemann sum is $f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$. Find a similar formula for the trapezoid rule for a function $f(x)$ on an interval $[a,b]$, using n subintervals. **(Don't try to use summation notation.)**

c) Show that the value computed using the trapezoid rule is actually just the average of the left Riemann sum and the right Riemann sum.

3. Not every function is Riemann integrable. The goal of this problem is to convince you that any *non-decreasing* function on the closed interval $[a,b]$ is integrable. A function f is non-decreasing if $x_1 < x_2$ implies that $f(x_1) \le f(x_2)$. That is, the function rises from left to right, or at least never falls.

For a non-decreasing function, the right Riemann sum for n subintervals is also the "upper sum," where the height of the n^{th} rectangle is given by the maximum value of the function on the subinterval. Any other Riemann sum for n subintervals will have a value that is the same or smaller than the upper sum. Similarly, for a non-decreasing function, the left Riemann sum for n subintervals is also the "lower sum" that has the smallest possible value for any Riemann sum with n subintervals. This is illustrated by the example from last week's lab:



a) Let $f(x)$ be a non-decreasing function on $[a, b]$. For a positive integer n, let U_n be the right Riemann sum for f using n subintervals, and let L_n be the corresponding left Riemann sum. Explain why $U_n - L_n = \frac{1}{n}(f(b) - f(a))$. This is just a generalization of problem 3 from Lab 1, and it can be proved in the same way. You could also give a geometric argument, with a picture! (So, L_n and U_n get closer and closer together as n increases. Also remember that any other Riemann sum for f with n subintervals has a value between L_n and U_n.)

b) Explain why $U_k \geq L_j$ even if $k \neq j$; that is, **any** right Riemann sum is greater than or equal to **any** left Riemann sum. Drawing a picture could help to justify your answer!

c) Explain why part **a)** and part **b)** together imply that f is integrable. That is, there is some number L such that as $n \to \infty$, all possible Riemann sums approach L. It can be helpful to draw a number line and mark sets of points U_n and L_n that satisfy **a)** and **b)**. Where is L on the number line? Why do U_n and L_n approach L as $n \to \infty$? Why do all the other possible Riemann sums also approach L?

(By the way, note that a small change to this argument would show that every *non-increasing* function on an interval $[a, b]$ is also integrable.)

4. The previous problem implies that some pretty strange functions are integrable. In particular, an integrable function can have an infinite number of discontinuities. For example, define a function $g(x)$ on $[0, 1]$ such that $g(x) = \frac{1}{2}$ for $\frac{1}{2} < x \leq 1$, $g(x) = \frac{1}{3}$ for $\frac{1}{3} < x \leq \frac{1}{2}$, $g(x) = \frac{1}{4}$ for $\frac{1}{4} < x \leq \frac{1}{3}$, and in general or any positive integer n, $g(x) = \frac{1}{n}$ for $\frac{1}{n} < x \leq \frac{1}{n-1}$. Finally, let $g(0) = 0$. This function is non-decreasing and therefor integrable.

a) Draw a picture to show, as well as you can, the graph of g.

b) Write down an infinite sum that represents the area under the graph of g, using the "$\cdots$" notation.

c) g is discontinuous at $x = \frac{1}{n}$ for every positive integer n. Do you think that g is continuous at $x = 0$? Why or why not?