Math 131-03 Sample answers to study guide questions

(1) a) $\sum_{k=1}^{\infty} \frac{k^3}{k^6+4}$ converges: $k^6+4 > k^6$, so $\frac{1}{k^6+4} < \frac{1}{k^6}$ and $\frac{k^3}{k^6+4} < \frac{k^3}{k^6} = \frac{1}{k^3}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series, $\sum_{k=1}^{\infty} \frac{k^3}{k^6+4}$ converges by the Comparison Test. (The Limit Comparison Test could also be used.)

b) $\sum_{k=1}^{\infty} \frac{k!}{7^k\sqrt{k}}$ diverges: Apply the Ratio Test: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$

$= \lim_{k\to\infty} \frac{(k+1)!}{7^{k+1}\sqrt{k+1}} \Big/ \frac{k!}{7^k\sqrt{k}} = \lim_{k\to\infty} \frac{(k+1)!\, 7^k\sqrt{k}}{k!\, 7^{k+1}\sqrt{k+1}}$

$= \lim_{k\to\infty} \frac{k+1}{7}\cdot\sqrt{\frac{k}{k+1}} = \infty$. Since this limit is greater than 1, the series diverges.

c) $\sum_{k=1}^{\infty} \left(-\frac{3}{5}\right)^k$ converges since it is a geometric series with $r = -\frac{3}{5}$. It converges because $|r|<1$.

d) $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{(k+1)^2}$ diverges by the Divergence Test since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{(-1)^k k^2}{(k+1)^2}$ does not exist. (Oscillates between ± 1.)

(2) $g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k\cdot 5^k}$. $g'(x) = \sum_{k=1}^{\infty} \frac{k x^{k-1}}{k\, 5^k} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{5^k}$.

So $g'(3) = \sum_{k=1}^{\infty} \frac{3^{k-1}}{5^k}$. This is a geometric series with $r = \frac{3}{5}$ and $a = \frac{1}{5}$, which converges to $\frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{3}{5}} = \frac{1}{2}$,

so $g'(3) = \underline{\underline{\frac{1}{2}}}$.

(3) $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$

$f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$, $f'''(1) = \frac{3}{8}$, so...

The Taylor polynomial is

$$f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3$$

$$=1+\frac{1}{2}(x-1)-\frac{1}{2}\cdot\frac{1}{4}(x-1)^2+\frac{1}{6}\cdot\frac{3}{8}(x-1)^3$$

$$=\underline{1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3}$$

(4) Since $\sum_{k=0}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k = 0$, which means $a_k < 1$ for large enough values of k, say for $k \geq M$. If $a_k < 1$, then $a_k^2 < a_k$ [Multiply both sides by a_k], for $k \geq M$. Since $\sum_{k=M}^{\infty} a_k$ converges, $\sum_{k=M}^{\infty} a_k^2$ also converges by the Comparison Test. But then $\sum_{k=1}^{\infty} a_k^2$ also converges.

[Alternative proof: $\lim_{k\to\infty}\frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$, so $\sum_{k=0}^{\infty} a_k^2$ converges by the Limit Comparison Test.]

(5) a) $\int x\sin(2x^2+1)\,dx$ — Simple substitution:

$w = 2x^2+1$, $dw = 4x\,dx$, so $\int x\sin(2x^2+1)\,dx$

$$=\frac{1}{4}\int \sin(w)\,dw = -\frac{1}{4}\cos(w)+C = \underline{-\frac{1}{4}\cos(2x^2+1)+C}$$

b) Integration by parts, $w = x$, $dv = \sin(2x+1)\,dx$
$dw = dx$, $v = -\frac{1}{2}\cos(2x+1)$

$$\int x\sin(2x+1)\,dx = wv - \int v\,dw = x\cdot\left(-\frac{1}{2}\cos(2x+1)\right) - \int -\frac{1}{2}\cos(2x+1)\,dx$$

$$= -\frac{1}{2}x\cos(2x+1) + \frac{1}{4}\sin(2x+1)+C$$

c) Simple substitution: $w = \sin(x^2)+2$, $dw = \cos(x^2)\cdot 2x\,dx$

$$\int \frac{2x\cos(x^2)}{\sin(x^2)+2}\,dx = \int \frac{1}{w}\,dw = \ln|w|+C = \underline{\ln(\sin(x^2)+2)+C}$$

d) partial fractions: $\frac{5}{(x-3)(2x-1)} = \frac{A}{x-3} + \frac{B}{2x-1}$;

$5 = A(2x-1) + B(x-3)$; $x = 3 \Rightarrow 5 = A(5) \Rightarrow A = 1$;

$x = \frac{1}{2} \Rightarrow 5 = B\cdot(\frac{1}{2}-3) \Rightarrow 5 = -\frac{5}{2}B \Rightarrow B = -2$; so

$\int \frac{5}{(x-3)(2x-1)}dx = \int \frac{1}{x-3} - \frac{2}{2x-1}dx = \underline{\ln|x-3| - \ln|2x-1| + C}$

(6) The curves $y = 4+x^2$ and $y = 2x^2$ intersect at $(2,8)$,

a) $\sum_{k=1}^{4} f(x_k^*) - g(x_k^*)$; $\Delta x = \frac{2-0}{4} = \frac{1}{2}$ and x_k^* are $0, \frac{1}{2}, 1, \frac{3}{2}$

The sum is $(4-0)\cdot\frac{1}{2} + ((4+(\frac{1}{2})^2) - 2\cdot(\frac{1}{2})^2)\cdot\frac{1}{2}$

$+ (5-2)\cdot\frac{1}{2} + ((4+(\frac{3}{2})^2) - 2(\frac{3}{2})^2)\cdot\frac{1}{2}$

This is an approximation for $\int_0^2 (4+x^2) - (2x^2)\,dx$

b) shell method: $\int_a^b 2\pi r h\,dx = \int_0^2 2\pi x(f(x) - g(x))\,dx$

$= \int_0^2 2\pi x((4+x^2) - (2x^2))\,dx$

c) washer method: $\int_a^b \pi(R^2 - r^2) = \int_0^2 \pi((4+x^2)^2 - (2x^2)^2)\,dx$

d) shell method: $\int_a^b 2\pi r h\,dx = \int_0^2 2\pi(x+2)(f(x) - g(x))\,dx$

$= \int_0^2 2\pi(x+2)((4+x^2) - (2x^2))\,dx$

(7) $a(t) = e^{-t}$. $v(t) = v_0 + \int_0^t a(x)\,dx = 0 + \int_0^t e^{-x}\,dx$

$= -e^{-x}\Big|_0^t = -e^{-t} + e^0 = 1 - e^{-t}$.

displacement $= \int_a^b v(t)\,dt = \int_0^2 1 - e^{-t}\,dt$

$= t + e^{-t}\Big|_0^2 = (2 + e^{-2}) - (0 + e^0)$

$= 2 + e^{-2} - 1 = \underline{1 + e^{-2}}$