Topological spaces

Math 135, Handout #1

In $\mathbb{R}$, the set of real numbers, we have the idea of an "open interval" such as $(1, 7)$, $(-\pi, \sqrt{2})$, or $(\frac{1}{2}, \infty)$. We can generalize this to the idea of "open set," which is defined as a union of open intervals. For example, $(1, 2) \cup (5, 7)$ is an open set. So is

$$\bigcup_{n=0}^{\infty} (3n, 3n + 1)$$

even though this is an infinite union. In fact, open sets in $\mathbb{R}$ have the property that the union of any collection of open sets is still an open set.

Note that the entire set of real numbers is an open set, since $\mathbb{R}$ is the open interval $(-\infty, \infty)$. What about the empty set, $\varnothing$? To make certain things work out nicely, the empty set is defined to be an open set. We can try to justify this by considering the empty set to be the union of zero open intervals. Or we could allow open interval notations such as $(1, 0)$; this would represent numbers that are greater than 1 and less than 0, which includes no numbers at all. But in any case, we say that $\varnothing$ is an open set.

It is also true that the intersection of two open sets in $\mathbb{R}$ is again an open set, and it follows that the intersection of any finite number of open sets in $\mathbb{R}$ is again an open set. (Note that for this to be true in all cases, the empty set must be open, since the intersection of two non-empty open sets can be empty.) However, it is possible that the intersection of an infinite collection of open sets is not an open set. For example,

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, \tfrac{1}{n} \right) = \{0\}$$

and the set $\{0\}$, which contains a single point, is not a union of open intervals.

There is another way to characterize open sets in $\mathbb{R}$. We start with a set of "basic" open sets, $(x - \epsilon, x + \epsilon)$, for $x \in \mathbb{R}$ and $\epsilon > 0$. A subset U of $\mathbb{R}$ is open if for every $x \in U$, there is some $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. Note that the ϵ depends on x. For example, for x in the open set $U = (0, 3)$, if x is very close to 0, then you need a very small ϵ to make sure that $(x - \epsilon, x + \epsilon)$ lies entirely inside U. but for $x = 2$, we could take ϵ to be as large as 1.

Finally, we note that there is also the idea of a "closed" subset of $\mathbb{R}$. By definition, a set is closed if and only if its complement is open. The complement $\overline{F}$, is taken in the set $\mathbb{R}$, that is, $\overline{F} = \mathbb{R} \smallsetminus F$. So F is a closed subset of $\mathbb{R}$ if and only if $\mathbb{R} \smallsetminus F$ is a union of open intervals (possibly an infinite union). For example, a closed interval $[a, b]$ is closed because it is the complement of the open set $(-\infty, a) \cup (b, \infty)$. But "closed" is not quite the opposite of "open." Many sets, such as $[1, 7)$ are neither open nor closed. And it is possible for a set to be both open and closed. For example, the empty set is open, but it is also closed since its complement is the entire set $\mathbb{R}$, which is open. It turns out that $\varnothing$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed.

In this handout, we look at a generalization of the idea of open and closed sets. We define a "topology" on a set X to be a collection of subsets of X that have properties similar to the open subsets of $\mathbb{R}$.

Definition 1. A **topological space** is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of subsets of X satisfying

1. $\varnothing \in \mathcal{T}$
2. $X \in \mathcal{T}$
3. Any union of sets in $\mathcal{T}$ is again a set in $\mathcal{T}$. (This includes infinite unions.) We say that $\mathcal{T}$ is closed under union.
4. Any intersection of a finite number of sets in $\mathcal{T}$ is again a set in $\mathcal{T}$. We say that $\mathcal{T}$ is closed under finite intersection.

The set $\mathcal{T}$ is called a **topology** on X. The sets in $\mathcal{T}$ are called the **open** sets in the topology.

Definition 2. Let $(X, \mathcal{T})$ be a topological space. A subset F of X is said to be **closed** in the topology $\mathcal{T}$ if and only if its complement, $X \setminus F$, is an open set in the topology.

To understand closed sets better, we need to understand how set complements work with unions and intersections. For that, we need DeMorgan's Laws for Sets, which say that the complement of a union of sets is the intersection of the complements of those sets, and the complement of an intersection of sets is the union of the complements of those sets. (Of course, for complements to be defined, we have to be working with subsets of some universal set.) For a pair of sets A and B, DeMorgan's Laws are stated as

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

but DeMorgan's laws are valid for any number of sets, even for an infinite family of sets. Suppose that $\{A_\alpha \,|\, \alpha \in \mathscr{A}\}$ is an indexed family of subsets of some universal set. Then

$$\overline{\bigcup_{\alpha \in \mathscr{A}} A_\alpha} = \bigcap_{\alpha \in \mathscr{A}} \overline{A_\alpha}$$
$$\overline{\bigcap_{\alpha \in \mathscr{A}} A_\alpha} = \bigcup_{\alpha \in \mathscr{A}} \overline{A_\alpha}$$

When we apply this to closed subsets in a topological space, we can prove the following theorem:

Theorem 1. Suppose that $(X, \mathcal{T})$ is a topological space. Then:

(1) $\varnothing$ is a closed subset of X;

(2) X is a closed subset of X;

(3) The intersection of any number, finite or infinite, of closed subsets of X is again a closed subset of X; and

(4) The union of any finite number of closed subsets of X is a again a closed subset of X.

Proof. (1) is true because $\varnothing$ is the complement of the open set X, and similarly (2) is true because X is the complement of the open set $\varnothing$.

To prove (3), suppose that $\{F_\alpha \,|\, \alpha \in \mathscr{A}\}$ is a collection of closed subsets of X. We need to show that the intersection of all of the F_α is a closed subset of X. By definition, a closed subset is the complement of an open subset, so we know that $\overline{F_\alpha}$ is open for all α. By the definition of topology, we know that the union of open sets is open, so $\bigcup_{\alpha\in\mathscr{A}} \overline{F_\alpha}$ is open. But then the complement of that set, $\overline{\bigcup_{\alpha\in\mathscr{A}} \overline{F_\alpha}}$, is closed. By DeMorgan's laws, that complement is equal to $\bigcap_{\alpha\in\mathscr{A}} \overline{(\overline{F_\alpha})}$ But clearly, $\overline{(\overline{F_\alpha})} = F_\alpha$, so we have shown that $\bigcap_{\alpha\in\mathscr{A}} F_\alpha$ is closed. That is, we have shown that intersection of any collection of closed subsets of X is closed.

(4) can be proved in the same way, using a finite family of closed subsets of X and the fact that a finite union of open sets is open. □

Now, suppose that we have a topological space $(X, \mathcal{T})$, and suppose that Y is a subset of X. We can then make Y into a topological space in a natural way: We define a subset A of Y to be open in the topology for Y if and only if there is an open set U in X such that $A = U \cap Y$. That is, we make the topological space $(Y, \mathcal{T}_Y)$ by defining

$$\mathcal{T}_Y = \{U \cap Y \,|\, U \in \mathcal{T}\}$$

When we do this, we say that Y is a **subspace** of X, and that Y has the **subspace topology**. We can show that a subset of Y is closed in Y if and only if it is the intersection of Y with a closed subset of X.

For example, lets consider the rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. To get an open set in $\mathbb{Q}$, we just start with an open set in $\mathbb{R}$ and take all the rational numbers in that open set. So, for example, the set $\{q \in \mathbb{Q} \,|\, 1 < q < 3\}$ is open in $\mathbb{Q}$ since it consists of the rational numbers in the open interval $(1, 3)$; that is, $\{q \in \mathbb{Q} \,|\, 1 < q < 3\}$ is equal to the intersection $(1, 3) \cap \mathbb{Q}$. Now consider the subset of $\mathbb{Q}$ given by

$$A = \{q \in \mathbb{Q} \,|\, q^2 < 2\} = \{q \in Q \,|\, -\sqrt{2} < q < \sqrt{2}\}$$

A is open in $\mathbb{Q}$ since it is the intersection of A with the open subset $(-\sqrt{2}, \sqrt{2})$ in $\mathbb{R}$. However, A is also closed in $\mathbb{Q}$ because it is the intersection of A with the closed subset $[-\sqrt{2}, \sqrt{2}]$ in $\mathbb{R}$. In fact, $\mathbb{Q}$ has many subsets that are both open and closed in the subspace topology.

* * *

Topology can be thought of as generalizing the idea of "nearness" or "closeness," but without ever bringing in the idea of distance. The idea is that a point y is somewhat close to a point x if y is in an open set that contains x. Of course, it could be a very large open set, in which case y is not necessarily all that close to x. However, if y is in most of the open sets that contain x, it could be considered to be pretty close to x. To be a little more precise, if U and V are open sets that contain x, and if $V \subseteq U$, then in some sense, the points of V can be thought of as being closer to x than points that are in $U \setminus V$. By taking smaller and smaller open sets that contain x, we get points that are closer and closer to x. We could use this idea to define convergence of sequences in terms of open sets without ever mentioning distance: A sequence $\{x_n\}_{n=1}^\infty$ converges to x if for any open set U that contains x, all the terms of the sequence after some point lie in U. More formally,

Definition 3. Let $(X, \mathcal{T})$ be a topological space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X. We say that $\{x_n\}_{n=1}^{\infty}$ **converges** to the point x in X if and only if for every open set U that contains x, there is a number N such that $x_n \in U$ for all $n \geq N$. If a sequence converges to some point, then we say that the sequence is **convergent**.

Convergence of sequences in $\mathbb{R}$ was already defined, using the idea of distance between points in $\mathbb{R}$. We should check that the new definition of convergence agrees with the old definition. According to the old definition, a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to x if for every $\epsilon > 0$, there is a number N such that $|x_n - x| < \epsilon$ for all $n \geq N$. But suppose that U is an open subset of $\mathbb{R}$ that contains x. Then there is an $\epsilon > 0$ such that the open interval $(x - \epsilon, x + \epsilon)$ is a subset of U. Furthermore, the interval $(x - \epsilon, x + \epsilon)$ contains exactly the points y such that $|y - x| < \epsilon$. So, if $|x_n - x| < \epsilon$ for all large n, then $x_n \in U$ for all large n. Using this, it's easy to see that the two definitions of convergence are the same.

Exercises

Exercise 1. Explain why a subset of $\mathbb{R}$ that consists of a single element, such as $\{2\}$, is a closed subset of $\mathbb{R}$. Based on that fact and properties of closed sets, explain why every finite subset of $\mathbb{R}$ is a closed subset of $\mathbb{R}$.

Exercise 2. Let X be any set. Let $\mathcal{T}$ be the the collection of **all** subsets of X (that is, $\mathcal{T} = \mathscr{P}(X)$). Explain why $\mathcal{T}$ is a topology for X. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X, and that the sequence converges to x. Show that there is a natural number N such that $x_n = x$ for all $n \geq N$.

Exercise 3. Let X be any infinite set. Let $\mathcal{F}$ be the collection of subsets of X consisting of the finite subsets of X. Is $\mathcal{F}$ a topology on X? (Explain!)

Exercise 4. Let X be any infinite set. Let $\mathcal{C}$ be the collection of subsets of X consisting of $\varnothing$ plus every set whose complement is finite. That is, $\mathcal{C} = \{\varnothing\} \cup \{U \subseteq X \mid X \smallsetminus U \text{ is finite}\}$. Is $\mathcal{C}$ a topology for X? (Explain!)

Exercise 5. Consider the closed interval $[1, 3]$ in $\mathbb{R}$ as a subspace of $\mathbb{R}$. Explain why the subset $[1, 2)$ of $[1, 3]$ is an open subset in the subspace $[1, 3]$, even though it is not an open subset of $\mathbb{R}$.

Exercise 6. If we consider the set of natural numbers, $\mathbb{N}$, as a subspace of $\mathbb{R}$, what are the open subsets of $\mathbb{N}$?

Exercise 7. Let $X = \mathbb{N} \cup \{0\}$. That is, X is the set of non-negative integers. In this exercise, we consider X with a rather strange topology. For each $i = 0, 1, 2, 3, \ldots$, define N_i to be the set $N_i = \{0, 1, 2, \ldots, i - 1\}$. So $N_o = \varnothing$, $N_1 = \{0\}$, $N_2 = \{0, 1\}$, $N_3 = \{0, 1, 2\}$, and so on. We can define a topology on X in which the open sets are precisely the sets $N_o, N_1, N_2, \ldots$, together with X itself. That is, we make the topological space $(X, \mathcal{T}\}$ where $\mathcal{T} = \{X, N_o, N_1, N_2, N_3, \ldots\}$. Show that $\mathcal{T}$ is in fact a topology. (What is the union of a family of sets in $\mathcal{T}$? What is the intersection of a family of sets in $\mathcal{T}$?) What are the closed subsets in this topological space?