Problem 1. The following questions about spans can be answered by solving linear systems of equations.

(a) Is the vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ in the span of the set $T = \left\{ \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ in $\mathbb{R}^3$?

(b) Is the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in the span of the set $T = \left\{ \begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\4\end{pmatrix}, \begin{pmatrix}-2\\2\\3\end{pmatrix} \right\}$ in $\mathbb{R}^3$?

(a) The question is whether there are scalars x and y such that

$$a \begin{pmatrix}-1\\0\\1\end{pmatrix} + b \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

This is equivalent to finding a solution of the linear system

$$\begin{aligned} -x + y &= 1 \\ y &= 2 \\ x + y &= 3 \end{aligned}$$

The second equation forces $y = 2$, and then the first equation becomes $-x + 2 = 1$, or $x = 1$. Note that $x = 1, y = 2$ is also a solution of the third equation. So the system as a whole as $x = 1, y = 2$ as a solution. This means that the given vector **is** in the span of T. (Note: It would probably be possible to find the solution by inspection and just check that it works in all three equations.)

(b) The question is whether there are scalars x, y, and x such that

$$x \begin{pmatrix}2\\-1\\1\end{pmatrix} + y \begin{pmatrix}0\\1\\4\end{pmatrix} + z \begin{pmatrix}-2\\2\\3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

This is equivalent to finding a solution of the linear system

$$\begin{array}{cccc} 2x & & -2z & = & 1 \\ -x & +y & +2z & = & 1 \\ x & +4y & +3z & = & 1 \end{array}$$

We can check this by putting the corresponding augmented matrix into echelon form:

$$\left(\begin{array}{ccc|c} 2 & 0 & -2 & 1 \\ -1 & 1 & 2 & 1 \\ 1 & 4 & 3 & 1 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 2 & 0 & -2 & 1 \\ 1 & 4 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[\rho_1+\rho_3]{2\rho_1+\rho_2} \left(\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 5 & 5 & 2 \end{array}\right)$$

$$\xrightarrow{-\frac{5}{2}\rho_2+\rho 3} \left(\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & -\frac{11}{2} \end{array}\right)$$

Since the last row is of the form $0 = k$, the system has no solution, which means that the given vector is **not** in the span of T.

Problem 2. Let S be the subset $T = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ of $\mathbb{R}^3$. Show that $[S]$, the span of S, is all of $\mathbb{R}^3$. (This is asking you to show that every $\vec{v} \in \mathbb{R}^3$ can be written as a linear combination of the vectors in S.)

The question is whether the following system can be solved for any $a, b, c \in \mathbb{R}$:

$$x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

The coefficient matrix for this system is

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

which is already in echelon form with no row of zeros. So, the coefficient matrix is non-singular, which means that the system always has exactly one solution.

Problem 3. Let $\mathscr{P}$ be the (infinite-dimensional) vector space of all polynomials. Let T be the subset of $\mathscr{P}$ given by $\{p_0(x), p_1(x), p_2(x), \dots\}$, where $p_0(x) = 1$, $p_1(x) = 1 + x$, $p_2(x) = 1 + x + x^2$, $p_3(x) = 1 + x + x^2 + x^3$, and, more generally, $p_n(x) = 1 + x + x^2 + \cdots + x^n$. Show that $[T]$, the span of T, is all of $\mathscr{P}$. (You just need to check that any polynomial $q(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ can be written as a linear combination of some finite number of elements of T. Hint: Problems 2 and 3 are almost the same question.)

Let $q(x) \in \mathscr{P}$. We need to write $q(x) = a_0p_0(x) + a_1p_1(x) + a_2p_2(x) + \cdots + a_kp_k(x)$ for some integer k and some $a_0, a_1, \ldots, a_k \in \mathbb{R}$. We can let k be the degree of $q(x)$, so $q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_kx^k$. Then, we need

$$\begin{aligned} b_0 + b_1x + b_2x^2 + \cdots + b_kx^k = a_0 &\cdot (1) \\ &+ a_1 \cdot (1+x) \\ &+ a_2 \cdot (1+x+x^2) \\ &+ a_3 \cdot (1+x+x^2+x^3) \\ &+ \cdots \\ &+ a_k \cdot (1+x+x^2+\cdots x^k) \end{aligned}$$

Equating te coefficients of x^i on the two sides of the equation for $i = 0, 1, \ldots, k$, we get a linear system with k equations and k variables:

$$\begin{aligned} a_0 + a_1 + a_2 + \cdots + a_k &= b_0 \\ a_1 + a_2 + \cdots + a_k &= b_1 \\ a_2 + \cdots + a_k &= b_2 \\ &\vdots \\ a_k &= b_k \end{aligned}$$

Here, the b_i are the given constant coefficients from $q(x)$, and the a_i are unknown. We must show that it is possible to solve this system to get values for $a_0, a_1, \ldots, a_k$. But the coefficient matrix for this system has the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

This is a square matrix in echelon form with no row of zeros, so it is non-singular, which implies that the system does have a solution.

Problem 4. Show that the following vectors in $\mathbb{R}^3$ are linearly dependent. (Recall that vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent if $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = 0$ for some $a, b, c \in \mathbb{R}$ where a, b, and c are not all zero.)

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

The vector equation that we need to consider is

$$a \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + c \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is eqivalent to this linear system in variables a, b, c:

$$\begin{array}{ccccc} 2a & & -2c & = & 0 \\ -a & +b & +2c & = & 0 \\ a & +4b & +3c & = & 0 \end{array}$$

Put the coefficient matrix into echelon form:

$$\begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & 2 \\ 1 & 4 & 3 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} -1 & 1 & 2 \\ 2 & 0 & -2 \\ 1 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow[\rho_1+\rho_3]{2\rho_1+\rho_2} \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \end{pmatrix}$$

$$\xrightarrow{-\frac{5}{2}\rho_2+\rho_3} \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This represents a homogeneous system with one free variable, which means that there is an infinite number of solutions. In particular, there is a non-zero solution. This means that the three vectors are linearly dependent.

Problem 5. Let $\mathscr{P}_3$ be the vector space of all polynomials of degree less than or equal to 3. Show that the following vectors in $\mathscr{P}_3$ are linearly independent: $1 - 2x$, $3x - 2x^2 + x^3$, $4 + x^2$.

To say that vectors are linearly independent means that if a linear combination of the vectors adds up to zero, then the coefficients in the linear combination must be zero. That is, we must show that if

$$a(1-2x) + b(3x - 2x^2 + x^3) + c(4 + x^2) = 0$$

then $a = b = c = 0$. Writing the left-hand side as a single polynomial gives

$$(a + 4c) + (-2a + 3b)x + (-2b + c)x^2 + b(x^3) = 0$$

which is true if and only if

$$\begin{array}{ccccc} a & & +4c & = & 0 \\ -2a & +3b & & = & 0 \\ & -2b & +c & = & 0 \\ & b & & = & 0 \end{array}$$

The last equation forces b to be zero. Plugging that into the second equation gives $c = 0$. And then plugging that into the first equation shows that a must also be zero. So, the only solution is $a = b = c = 0$, which means that the three polynomials are linearly independent.