Problem 1. Some of the following matrix products are not defined. For each product, you should either compute the product, if it is defined, or state why it is not defined.

a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & -1 \\ -2 & 2 \\ 1 & 0 \end{pmatrix}$ **b)** $\begin{pmatrix} 0 & 3 & 5 \\ 2 & 1 & 7 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **d)** $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 \\ 5 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & -1 \\ -2 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1+6-6+4 & 0-2+6+0 \\ -4+9-4+1 & 0-3+4+0 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

(b) The product is not defined because the number of columns in the first matrix (3) is not equal to the number of rows in the second matrix (2).

(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0+c & 0+d \\ a+0 & b+0 \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

(d) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0+b & a+0 \\ 0+d & c+0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

(e) The product is not defined because the number of columns in the first matrix (1) is not equal to the number of rows in the second matrix (2).

Problem 2. Let $A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$. Note that $A \in M_{4\times 1}$ and $B \in M_{1\times 4}$. Compute the matrix products AB and BA.

$$AB = \begin{pmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 & 1\cdot 4 \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & 2\cdot 4 \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & 3\cdot 4 \\ 4\cdot 1 & 4\cdot 2 & 4\cdot 3 & 4\cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

$$BA = (1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4) = (30)$$

Problem 3. Find the inverse of each of the following matrices, or show that the matrix has no inverse. For part (c), you should find the inverse by putting the augmented matrix

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right)$$

into reduced echelon form.

a) $\begin{pmatrix} 3 & -2 \\ 5 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$ **c)** $\begin{pmatrix} -1 & 3 & 0 \\ 2 & -1 & 5 \\ 1 & 2 & -5 \end{pmatrix}$

(a) Writing $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 5 & 4 \end{pmatrix}$, we have $ad - bc = 22$, and the inverse is

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 4 & 2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2/11 & 1/11 \\ -5/22 & 3/22 \end{pmatrix}$$

(b) Here, $ad - bc = 0$ so there the matrix has no inverse.

(c)

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\rho_1+\rho_3]{2\rho_1+\rho_2} \left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 5 & 5 & 2 & 1 & 0 \\ 0 & 5 & -5 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 5 & 5 & 2 & 1 & 0 \\ 0 & 0 & -10 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\substack{-\rho_1 \\ \frac{1}{5}\rho_2 \\ -\frac{1}{10}\rho_3}} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1 & 1/10 & 1/10 & -1/10 \end{array}\right)$$

$$\xrightarrow{-\rho_3+\rho_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3/10 & 1/10 & 1/10 \\ 0 & 0 & 1 & 1/10 & 1/10 & -1/10 \end{array}\right)$$

$$\xrightarrow{3\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/10 & 3/10 & 3/10 \\ 0 & 1 & 0 & 3/10 & 1/10 & 1/10 \\ 0 & 0 & 1 & 1/10 & 1/10 & -1/10 \end{array}\right)$$

So the inverse is

$$\begin{pmatrix} -1/10 & 3/10 & 3/10 \\ 3/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & -1/10 \end{pmatrix}$$

Problem 4. We can define the derivative for all polynomials as a homomorphism $D\colon \mathscr{P} \to \mathscr{P}$. The null space $\mathscr{N}(D)$ is the set of all constant polynomials. We can form the composition $D \circ D$, which also maps $\mathscr{P}$ to $\mathscr{P}$. For a polynomial $q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, what is $D \circ D(q(x))$? What mathematical operation does $D \circ D$ compute? What is the null space of $D \circ D$?

$D \circ D$ represents the second derivative, since $D \circ D(q(x)) = D(D(q(x))) = D(q'(x)) = q''(x)$. For the polynomial $q(x)$ as given in the problem,

$$\begin{aligned} D(D(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)) &= D(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^n) \\ &= 2a_2 + 6a_3 x + \cdots n(n-1)a^n \end{aligned}$$

We can see that $D \circ D(q(x)) = 0$ if and only if $a_3 = a_4 = \cdots = 0$. This is true if and only if $q(x) = a_0 + a_1 x$ for some a_0, a_1, that is, if and only if $q(x)$ has degree less than 2.

Problem 5. Suppose that V and W are vector spaces and that $f\colon V \to W$ is a homomorphism. Suppose that $S \subseteq V$ and that S spans V. Show that the set $f(S)$ spans the range space $\mathscr{R}(f)$ of f. (Note: $\mathscr{R}(f) = \{f(\vec{v}) \mid \vec{v} \in V\}$, and $f(S) = \{f(\vec{v}) \mid \vec{v} \in S\}$.)

Let $\vec{w} \in \mathscr{R}(f)$. We must show that $\vec{w}$ is equal to a linear combination of elements of $f(S)$. Since $w \in \mathscr{R}(f)$, there is a $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$. Since S spans V, we can write $\vec{v} = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots c_k\vec{s}_k$ for some $s_1, s_2, \ldots, s_k \in S$. But then $\vec{w} = f(\vec{v}) = f(c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots c_k\vec{s}_k) = c_1 f(\vec{s}_1) + c_2 f(\vec{s}_2) + \cdots c_k f(\vec{s}_k)$. Since $f(\vec{s}_1), \ldots, f(\vec{s}_k) \in f(S)$, we have written $\vec{w}$ as a linear combination of elements of $f(S)$.

Problem 6. Suppose that V and W are vector spaces and that $f\colon V \to W$ is a homomorphism. Assume that h is one-to-one (which implies that the null space $\mathscr{N}(h)$ contains only $\vec{0}_V$). And suppose that $\langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \rangle$ is a linearly independent sequence of vectors in V. Show that the sequence $\langle f(\vec{v}_1), f(\vec{v}_2), \ldots, f(\vec{v}_k) \rangle$ is linearly independent (in W).

Suppose that $c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) + \cdots + c_k f(\vec{v}_k) = \vec{0}$. We must show that $c_1 = c_2 = \cdots = c_k = 0$. But $c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) + \cdots + c_k f(\vec{v}_k) = f(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k)$. Since this is $\vec{0}$, we have that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \in \mathscr{N}(f)$. Since the null space is just $\{\vec{0}\}$, we see that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$. Then, since $\langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \rangle$ is linearly independent, we must have $c_1 = c_2 = \cdots = c_k = 0$.