

Four linear systems consisting of four equations in three variables:

$$\begin{array}{rcl} x + 2y - z & = & 2 \\ y + z & = & 3 \\ z & = & 1 \\ 0 & = & 7 \end{array}$$

$$\begin{array}{rcl} x + 2y - z & = & 2 \\ y + z & = & 3 \\ z & = & 1 \\ 0 & = & 0 \end{array}$$

$$\begin{array}{rcl} x + 2y - z & = & 2 \\ y + z & = & 3 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}$$

$$\begin{array}{rcl} x + 2y - z & = & 2 \\ z & = & 1 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}$$

Solutions:

NO SOLUTION

$$\begin{array}{l} z = 1 \\ y = 3 - z \\ \quad = 2 \\ x = 2 - 2y + z \\ \quad = -1 \end{array}$$

$$\begin{array}{l} z \text{ is free} \\ y = 3 - z \\ x = 2 - 2y + z \\ \quad = -4 + 3z \end{array}$$

$$\begin{array}{l} z = 1 \\ y \text{ is free} \\ x = 2 - 2y + z \\ \quad = 2 - 2y + 1 \\ \quad = 3 - 2y \end{array}$$

Solution sets:

$$\emptyset \quad \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\begin{aligned} & \left\{ \begin{pmatrix} -4 + 3z \\ 3 - z \\ z \end{pmatrix} : z \in \mathbb{R} \right\} \\ & = \left\{ \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \end{aligned}$$

$$\begin{aligned} & \left\{ \begin{pmatrix} 3 - 2y \\ y \\ 1 \end{pmatrix} : y \in \mathbb{R} \right\} \\ & = \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} : y \in \mathbb{R} \right\} \end{aligned}$$

And a system with two free variables...

$x + 2y - z = 2$, so $x = 2 - 2y + z$ where y and z are free

$$\text{Solution set: } \left\{ \begin{pmatrix} 2 - 2y + z \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : y, z \in \mathbb{R} \right\}$$