Problem 1 (Exercise 1.1.12)**.** Prove that if a is irrational, then $\sqrt{a}$ is also irrational.

Answer:

We prove the contrapositive, which is equivalent. That is, we prove: If $\sqrt{a}$ is not irrational, then a is not irrational. That is, if $\sqrt{a}$ is rational, then a is rational.

So, assume $\sqrt{a}$ is rational. We want to show that a is rational. Since $\sqrt{a}$ is rational, then $\sqrt{a} = \frac{k}{n}$ where k and n are integers and $n \neq 0$. But then $a = (\sqrt{a})^2 = \left(\frac{k}{n}\right)^2 = \frac{k^2}{n^2}$. This proves that a is rational, because k^2 and n^2 are integers and $n^2 \neq 0$.

[A proof by contradiction is also possible: Let a be irrational. Suppose, for the sake of contradiction that $\sqrt{a}$ is rational. . .]

Problem 2 (Exercises 1.1.14)**.** Show that $\sqrt{3}+\sqrt{2}$ is irrational as follows: First, show that if $\sqrt{3}+\sqrt{2}$ is rational then so is $\sqrt{3}-\sqrt{2}$. (Hint: Consider their product.) Second, show that $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ cannot both be rational. (Hint: Consider their sum.)

Answer:

Assume that $\sqrt{3}+\sqrt{2}$ is rational. Note that $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$. So, $\sqrt{3}-\sqrt{2} = \frac{1}{\sqrt{3}+\sqrt{2}}$. Then, since $\sqrt{3}+\sqrt{2}$ is rational, and the reciprocal of a non-zero rational number is rational, we have that $\sqrt{3}-\sqrt{2}$ is rational. We have shown that if $\sqrt{3}+\sqrt{2}$ is rational, then so is $\sqrt{3}-\sqrt{2}$.

Next we show that $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ cannot both be rational. By the previous result, this will show that $\sqrt{3}+\sqrt{2}$ is not rational (since, if it were, then both numbers would be rational).

Now, assume, for the sake of contradiction, that $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ are both rational. Since the sum of two rational numbers is rational, $(\sqrt{3}+\sqrt{2}) + (\sqrt{3}-\sqrt{2})$ is also rational. But that sum is equal to $2\sqrt{3}$, which is not rational. This contradiction shows that $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ cannot both be rational. [Note: If $2\sqrt{3}$ were rational, then $\frac{2*\sqrt{3}}{2}$ would also be rational. But we know that $\sqrt{3}$ is not rational.]

Problem 3. Determine whether each set is bounded above and if so find its least upper bound. Remember to briefly explain your answers. For D and E, you will need to quote some well-know facts about the relevant infinite series.

$A = \{1 - \frac{1}{n} \,|\, n \in \mathbb{N}\}$

$B = \{1 + \frac{1}{n} \,|\, n \in \mathbb{N}\}$

$C = [2, 9)$

$D = \{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \ldots\}$

$E = \{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \ldots\}$

Answer:

A) The least upper bound is 1. Since $1-\frac{1}{n}<1$ for all $n\in\mathbb{N}$, 1 is an upper bound. For any $x<1$, x is not an upper bound, since there is an $n_o\in\mathbb{N}$ with $1-\frac{1}{n_o}>x$. So, 1 is the least upper bound. [To find n_o, just choose any $n_o>\frac{1}{1-x}$.]

B) The least upper bound is 2. We can write $B=\{1+\frac{1}{2},1+\frac{1}{3},1+\frac{1}{4},\dots\}$. The sequence is decreasing, so the maximum is the first element, 2.

C) The least upper bound is 9. 9 is an upper bound, and any number α less than 9 is not an upper bound, since there are numbers in $[2,9)$ greater than α. So 9 is the least upper bound.

D) The least upper bound is 2. The sum $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$ is a partial sum of the geometric series $\sum_{k=0}^{\infty}\frac{1}{2^k}$, which has sum 2. All the partial sums are less than 2, so 2 is an upper bound for D, and the partial sums get arbitrarily close to 2, so 2 is the least upper bound.

E) The set is not bounded above. The sum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$ is a partial sum of the harmonic series, which diverges to infinity.

Problem 4 (From exercise 1.2.6)**.** Let A and B be arbitrary non-empty, bounded-above sets of real numbers. Define $C=\{a+b\,|\,a\in A \text{ and } b\in B\}$. [That is, C contains contains all sums made up of one number from A and one number from B.]

(a) Suppose that μ_1 is an upper bound for A and μ_2 is an upper bound for B. Let $\mu=\mu_1+\mu_2$. Show that is an upper bound for C.

(b) Now suppose that λ_1 is the least upper bound for A and λ_2 is the least upper bound for B. Let $\lambda=\lambda_1+\lambda_2$. Show that λ is the least upper bound for C. (Hint: Use the last theorem in the third reading guide: Let $\varepsilon>0$. Explain why there is an $a_o\in A$ such that $a_o>\lambda_1-\frac{\varepsilon}{2}$ and a $b_o\in B$ such that $b_o>\lambda_2-\frac{\varepsilon}{2}$. Use this to show $a_o+b_o>\lambda-\varepsilon$, and conclude that λ is the least upper bound for C.)

Answer:

(a) Let $c\in C$. We want to show that $c\le\mu$, where $\mu=\mu_1+\mu_2$. Since $c\in C$, there is an $a\in A$ and a $b\in B$ such that $c=a+b$. We know that $a\le\mu_1$ and $b\le\mu_2$. It follows that $a+b\le\mu_1+\mu_2$. That is, $c<\mu$.

(b) The theorem says that an upper bound x is the least upper bound of the set X if and only if for every $\varepsilon>0$, there is a $y\in X$ with $y>x-\varepsilon$. By (a), λ is an upper bound for C. We what to show that λ is the least upper bound. So let $\varepsilon>0$. We must find $c_o\in C$ with $c_o>\lambda-\varepsilon$. Since λ_1 is the least upper bound of A and $\frac{\varepsilon}{2}>0$, there is an $a_o\in A$ with $a_o>\lambda_1-\frac{\varepsilon}{2}$. Since λ_2 is the least upper bound of B and $\frac{\varepsilon}{2}>0$, there is a $b_o\in B$ with $b_o>\lambda_2-\frac{\varepsilon}{2}$. Adding these two inequalities gives $a_o+b_o<\lambda_1+\lambda_2-\varepsilon=\lambda-\varepsilon$. Let $c_o=a_o+b_o$, which is is C. So, $c_o>\lambda-\varepsilon$, as we wanted to show.

Problem 5 (From exercise 1.2.4)**.** Consider two sequences of real numbers $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$, which are bounded above. Let C be the set $C = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots\}$. [Compare this to the previous problem, where C contains only the sums of all elements of A with all elements of B; the C in this problem contains only sums of corresponding elements from the two sequences.]

(a) Suppose that μ_1 is an upper bound for A and μ_2 is an upper bound for B. Show that $\mu_1 + \mu_2$ is an upper bound for C.

(b) Now suppose that λ_1 is the least upper bound for A and λ_2 is the least upper bound for B. Give an example to show that $\lambda_1 + \lambda_2$ is not necessarily the least upper bound of C. [Hint: Take part (c) into account as you look for an example!]

(c) Show that if A and B are *non-decreasing* sequences, then λ is in fact the least upper bound of C. (Non-decreasing here means $a_1 \le a_2 \le a_3 \le \cdots$ and $b_1 \le b_2 \le b_3 \le \cdots$.)

Answer:

(a) Let $c \in C$. We want to show that $c \le \mu$, where $\mu = \mu_1 + \mu_2$. Since $c \in C$, there is an $i \in \mathbb{N}$ such that $c = a_i + b_i$. We know that $a_i \le \mu_1$ and $b_i \le \mu_2$. It follows that $a_i + b_i \le \mu_1 + \mu_2$. That is, $c < \mu$.

(b) Let $A = \{1, 0, 0, 0, \dots\}$ and $B = \{0, 1, 1, 1, 1, \dots\}$. Then $\lambda_1 = 1$ is the least upper bound of A and $\lambda_2 = 1$ is also the least upper bound of B. So in this example, $\lambda_1 + \lambda_2 = 2$. However, $C = \{1, 1, 1, 1, 1, \dots\}$, which has least upper bound 1. So, $\lambda_1 + \lambda_2$ is not the least upper bound of C. [Another example: $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ and $B = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.]

(c) Let $\varepsilon > 0$. To show that λ is the least upper bound of C, we must find $c_o \in C$ with $c_o > \lambda - \varepsilon$. Since λ_1 is the least upper bound of A and $\frac{\varepsilon}{2} > 0$, there is an $i \in \mathbb{N}$ with $a_i > \lambda_1 - \frac{\varepsilon}{2}$. Since λ_2 is the least upper bound of B and $\frac{\varepsilon}{2} > 0$, there is a $j \in \mathbb{N}$ with $b_j > \lambda_2 - \frac{\varepsilon}{2}$. In the case $i = j$, we can simply let $c_o = a_i + b_i$ and add the two inequalities to get $c_o = a_i + b_i < \lambda_1 - \frac{\varepsilon}{2} + \lambda_2 - \frac{\varepsilon}{2} = \lambda - \varepsilon$. Next, consider the case $i < j$. Since the sequence $\{a_1, a_2, a_3, \dots\}$ is non-decreasing, and $i < j$, we know that $a_j \ge a_i$. From that and $a_i > \lambda_1 - \frac{\varepsilon}{2}$. we get $a_j > \lambda_1 - \frac{\varepsilon}{2}$. If we combine this inequality with $b_j > \lambda_2 - \frac{\varepsilon}{2}$, we get $a_j + b_j > \lambda_1 + \lambda_2 - \varepsilon$. So, in the case $i < j$, we can take $c_0 = a_j + b_j$. Similarly, in the final case, $j > i$, we can use $c_o = a_i + b_i$.

Problem 6. The last theorem in the third reading guide is about least upper bounds. State the corresponding theorem for greatest lower bounds. You do not have to prove the theorem.

Answer:

Theorem: Let X be a non-empty subset of $\mathbb{R}$ that is bounded below, and let μ be a lower bound for X. Then μ is the greatest lower bound of X if and only if for every $\varepsilon > 0$, there is a $y \in X$ such that $y < \mu + \varepsilon$.

Problem 7 (Exercises 1.2.17 aamd 1.2.18).

(a) Prove that the intersection of two Dedekind cuts is again a Dedekind cut.

(b) Show that the intersection of an infinite number of Dedekind cuts is not necessarily a Dedekind cut, even if the intersection is non-empty, by using the following example: For $n \in \mathbb{N}$, let S_n be the Dedekind cut corresponding to the number $\frac{1}{n}$. You need to show that $\bigcap_{n=1}^{\infty} S_n$ is not a Dedekind cut.

Answer:

(a) Suppose that α and β are Dedekind cuts. The second reading guide proved the trichotomy law, which says that exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$ is true. By definition, this means one of $\alpha \subset \beta$, $\alpha = \beta$, or $\alpha \supset \beta$ is true. Now, $\alpha \cap \beta = \alpha$ in the cases $\alpha \subset \beta$ or $\alpha = \beta$, and $\alpha \cap \beta = \beta$ in the case when $\alpha \supset \beta$. So, in any case, $\alpha \cap \beta$ is a Dedekind cut.

Alternative direct proof: Suppose that α and β are Dedekind cuts. Let $\gamma = \alpha \cap \beta$. We must show that γ satisfies the three properties of a Dedekind cut.

(i) γ is not empty and $\gamma \neq \mathbb{Q}$: Let $p \in \alpha$ and $q \in \beta$. Suppose $p \leq q$. Since β is a Dedekind cut, then $p \in \beta$ by property ii of Dedekind cuts. So, in this case, $p \in \alpha \cap \beta$. Similarly, if $q \leq p$, then $q \in \alpha \cap \beta$. So $\alpha \cap \beta$ contains either p or q and so is not empty. Since $\alpha \cap \beta \subseteq \alpha \subset \mathbb{Q}$, $\alpha \cap \beta$ is not $\mathbb{Q}$,

(ii) If $p \in \gamma$ and $q < p$, then $q \in \gamma$: Suppose that $p \in \gamma$ and $q < p$. Since α is a Dedekind cut, $q \in \alpha$, and since β is a Dedekind cut, $q \in \beta$. It follows that $q \in \alpha \cap \beta$.

(iii) If $p \in \gamma$, there is a $r \in \gamma$ such that $r > p$: Suppose that $p \in \gamma$. Since $p \in \alpha$, there is an $r_1 \in \alpha$ with $r_1 > p$. Since $p \in \beta$, there is an $r_2 \in \beta$ with $r_2 > p$. Consider the case $r_1 \leq r_2$; the case $r_2 \leq r_1$ is similar. Since $r_1 \leq r_2$, and $r_2 \in \beta$, then $r_1 \in \beta$. If we let $r = r_1$, then r is both in α and in β, and therefore is in their intersection, γ.

(b) In fact, $S_n = \{q \in \mathbb{Q} \,|\, q < \frac{1}{n}\}$. Since $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$, then $0 \in \bigcap_{n=1}^{\infty} S_n$. We show that S_n does not satisfy property iii of Dedekind cuts. That is, we show that there is no $r > 0$ such that $r \in \bigcap_{n=1}^{\infty} S_n$. For let $r \in \mathbb{Q}$ with $r > 0$. Now, there is an $n_o \in \mathbb{N}$ such that $\frac{1}{n_o} < r$. (For example, if $r = \frac{a}{b}$ where a and b are positive integers, let $n_o = b + 1$, so $\frac{1}{b+1} < \frac{1}{b} \leq \frac{a}{b}$.) But then $r \notin S_{n_o}$, and so $r \notin \bigcap_{n=1}^{\infty} S_n$.