Problem 1. Let (M, d) be a metric space, and let A be a subset of M. Prove that A is open if and only if A is equal to a union of open balls.

Answer:

$\Longrightarrow)$ Suppose that A is an open set in the metric space (M, d). Then for any $x \in A$, there is an $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq A$. We claim that $\displaystyle\bigcup_{x\in A} B_{\varepsilon_x}(x) = A$. But since $B_{\varepsilon_x}(x) \subseteq A$, it is also true that $\displaystyle\bigcup_{x\in A} B_{\varepsilon_x}(x) \subseteq A$. And for $a \in A$, we have $a \in B_{\varepsilon_x}(a) \subseteq \displaystyle\bigcup_{x\in A} B_{\varepsilon_x}(x)$, so $A \subseteq \displaystyle\bigcup_{x\in A} B_{\varepsilon_x}(x)$.

$\Longleftarrow)$ Suppose that A is a union of open balls, say $A = \displaystyle\bigcup_{\alpha\in B} B_{\varepsilon_\alpha}(x_\alpha)$. Since $B_{\varepsilon_\alpha}(x_\alpha)$ is an open set, A is a union of open sets, and we know that any union of open sets is open.

Problem 2. Consider the metric space $(\mathbb{R}, d)$, where d is the usual metric on $\mathbb{R}$ For each $n = 1, 2, 3, \dots$, let $\mathcal{O}_n$ be the open set $\mathcal{O}_n = \big(1, 1 + \frac{1}{n}\big)$. Show that $\{\mathcal{O}_n \,|\, n = 1, 2, \dots\}$ is an infinite collection of open sets whose intersection is not open. And find an infinite collection of closed subsets of $(\mathbb{R}, d)$ whose union is not closed.

Answer:

In fact, the intersection of all the $\mathcal{O}_n$ is the empty set, which is open. So the problem is incorrect.

If $\mathcal{O}_n = \big(1 - \frac{1}{n}, 1 + \frac{1}{n}\big)$, then $\displaystyle\bigcap_{n=1}^{\infty} \mathcal{O}_n = \{1\}$, because 1 is a member of each of the intervals, and for any number $x > 1$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < x - 1$, and therefore $1 + \frac{1}{n} < x$ and $x \notin \big(1 - \frac{1}{n}, 1 + \frac{1}{n}\big)$, and similarly, if $x < 0$, then x is not in the intersection. The set $\{1\}$ is not open because it does not contain any open ball about 1.

For another example, if $\mathcal{O}_n = \big(0, 1 + \frac{1}{n}\big)$, then the intersection is $(0, 1]$, which is not open.

For an infinite collection of closed sets whose union is not closed, we can use $\displaystyle\bigcup_{x\in(0,1)} \{x\}$. We have shown that any singleton set, $\{x\}$, is closed. This union is clearly equal to the interval $(0, 1)$, which is not closed since it does not include 1, which is an accumulation point of the set. (For a more traditional example, use $\displaystyle\bigcup_{n=1}^{\infty} \big[0, 1 - \frac{1}{n}\big]$. This union is equal to $[0, 1)$, which is not closed.)

Problem 3. Let X be any non-empty, bounded subset of $\mathbb{R}$, and let λ be the least upper bound of X. Show that $\lambda \in \overline{X}$. That is, the least upper bound of any set is an element of the closure of that set. [Hint: Use the definition of closure of X as the set of all points of X plus all accumulation points of X, and use Problem 1 from Homework 3.]

Answer:

Let X be a bounded non-empty subset of $\mathbb{R}$, and let $\lambda = lub(X)$. We want to show $\lambda \in \overline{X}$. The closure, $\overline{X}$, is defined to be the set of all points of X plus all accumulation points of X. We consider two cases: either $\lambda \in X$ or $\lambda \notin X$. In the case $\lambda \in X$, then $\lambda \in \overline{X}$ by definition. Consider the case $\lambda \notin X$. We have shown that when the least upper bound of a set is not a member of the set, then it is an accumulation point of that set. So in this case, λ is an accumulation point of X, and that means $\lambda \in \overline{X}$ by definition.

Problem 4. Let (A, σ), (B, τ), and (C, η) be metric spaces. Let $f\colon A \to B$ and $g\colon B \to C$. Suppose that f and g are continuous functions. Prove that their composition, $g \circ f$, is a continuous function.

Answer:

Let (A, σ), (B, τ), and (C, η) be metric spaces. Let $f\colon A \to B$ and $g\colon B \to C$. Suppose that f and g are continuous functions. We want to show that $g \circ f$ is continuous. Let $a \in A$, and let $\varepsilon > 0$. We want to find $\delta > 0$ such that $B_\delta(a) \subseteq B_\varepsilon(g(f(a)))$.

Since g is continuous at $f(a)$, there is a $\eta > 0$ such that $B_\eta(f(a)) \subseteq B_\varepsilon(g(f(a)))$. Since f is continuous at a, there is a $\delta > 0$ such that $B_\delta(a) \subseteq B_\eta(f(a)$. So, we have that $B_\delta(a) \subseteq B_\eta(f(a)) \subseteq B_\varepsilon(g(f(a)))$, and therefore for this δ, $B_\delta(a) \subseteq B_\varepsilon(g(f(a)))$.

(Alternative proof: Let $\mathcal{O}$ be open in C. We want to show $(g \circ f)^{-1}(\mathcal{O}\}$ is open in A. Now, $(g \circ f)^{-1}(\mathcal{O}\} = f^{-1}\big(g^{-1}(\mathcal{O})\big)$. Since g is continuous, $g^{-1}(\mathcal{O})$ is open in B. Since f is continuous, $f^{-1}\big(g^{-1}(\mathcal{O})\big)$ is open in A. So we are done.)

(Another alternative proof: Let $\{x_n\}_{n=1}^\infty$ converge to a in A. We want to show $\{g(f(x_n))\}_{n=1}^\infty$ converges in C to $g(f(a))$. Since f is continuous, $\{f(x_n)\}_{n=1}^\infty$ converges to $f(a)$ in B. Then since g is continuous, $\{g(f(x_n))\}_{n=1}^\infty$ converges to $g(f(a))$ in C. So we are done.)

Problem 5. Let $\{x_n\}_{n=1}^\infty$ be a convergent sequence in a metric space. Show that its limit is unique. That is, prove the following statement: if $\{x_n\}_{n=1}^\infty$ converges to y and $\{x_n\}_{n=1}^\infty$ converges to z, then $y = z$.

Answer:

Suppose that $\lim\limits_{n\to\infty} x_n = x$ and also $\lim\limits_{n\to\infty} x_n = y$. We want to show $x = y$. Suppose, for the sake of contradiction, that $x \neq y$.

Let $\varepsilon = d(x, y)/2$, which is greater than zero since $x \neq y$. There is an $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, $d(x_n, x) < \varepsilon$. And there is an $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, $d(x_n, y) < \varepsilon$. Let $N = \max(N_1, N_2)$. We then have both $d(x_N, x) < \varepsilon$ and $d(x_N, y) < \varepsilon$. But that means that

$$d(x, y) \leq d(x, x_N) + d(x_N, y) < \varepsilon + \varepsilon = d(x, y)/2 + d(x, y)/2 = d(x, y)$$

The contradiction $d(x, y) < d(x, y)$ proves that $x \neq y$ cannot be the case.

(Alternative direct proof: Show that $d(x, y) < \varepsilon$ for all $\varepsilon > 0$, which will prove $d(x, y) = 0$ and hence $x = y$. Let $\varepsilon > 0$. There is an $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, $d(x_n, x) < \frac{\varepsilon}{2}$. And there is an $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, $d(x_n, y) < \frac{\varepsilon}{2}$. Let $N = \max(N_1, N_2)$. Then $d(x, y) \leq d(x, x_N) + d(x_N, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.)