Problem 1. A previous homework problem showed that $\frac{1}{2}x|x|$ is an antiderivative for $|x|$. Using that fact, evaluate $\int_{-3}^{5} |x|\, dx$ **using the first Fundamental Theorem of Calculus once**. Explain why the answer makes sense geometrically (in terms of area).

Answer:

Let $f(x) = |x|$ and $g(x) = \frac{1}{2}x|x|$. We know that $g'(x) = f(x)$, so by the First Fundamental Theorem of Calculus, $\int_{-3}^{5} |x|\, dx = \int_{-3}^{5} f(x)\, dx = g(5) - g(-3) = \frac{1}{2} \cdot 5 \cdot |5| - \frac{1}{2} \cdot (-3) \cdot |-3| = \frac{1}{2} \cdot 25 + \frac{1}{2} \cdot 9 = 17$.

This makes since geometrically since the area under $y = |x|$ from -3 to 5 consists of two right triangles. One of the triangles has base and height equal to 3, and so area $\frac{9}{2}$, and the other has base and height equal to 5, and so area $\frac{25}{2}$. The total area is 17.

Problem 2. (a) Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series, and $\sum_{k=1}^{\infty} b_k$ is a divergent series. Show that $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges. [Hint: Proof by contradiction will work.]

(b) Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are both divergent. Give a simple example to show that $\sum_{k=1}^{\infty}(a_k + b_k)$ does not necessarily diverge.

Answer:

(a) Suppose for the sake of contradiction that $\sum_{k=1}^{\infty}(a_k + b_k)$ convergs. Then, since $\sum_{k=1}^{\infty} a_k$ also converges, we have by linearity that $\sum_{k=1}^{\infty} \big((a_k + b_k) + (-1) * a_k\big)$ converges. But in fact, $\sum_{k=1}^{\infty} \big((a_k + b_k) + (-1) * a_k\big) = \sum_{k=1}^{\infty} b_k$, which diverges. This contradiction proves the result.

(b) Let $\sum_{k=1}^{\infty} a_k$ be the divergent p-series $\sum_{k=1}^{\infty} \frac{1}{k}$, and $\sum_{k=1}^{\infty} b_k$ be the series $\sum_{k=1}^{\infty} \frac{-1}{k}$, which also diverges [since if it converged, so would $(-1)\sum_{k=1}^{\infty} \frac{-1}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$]. But we see that $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} 0$, which converges to 0.

Problem 3. A nonnegative series must either converge (absolutely) or diverge to $+\infty$. Classify each of the following nonnegative series as either convergent or divergent. In all cases, explain your reasoning, being explicit about any convergence tests that you apply.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2}{7k^5 + 2k^2}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2 + 1}}$ **c)** $\displaystyle\sum_{n=0}^{\infty} \pi^{-n}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

e) $\displaystyle\sum_{m=1}^{\infty} \frac{1 + \sin(m)}{5^m}$ **f)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^6}{5^n}$ **g)** $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ **h)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

Answer:

(a) Convergent. $\frac{3k^2}{7k^5+2k^2} < \frac{3k^2}{9k^5} = \frac{1}{3} \cdot \frac{1}{k^3}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series, $\sum_{k=1}^{\infty} \frac{1}{3}\frac{1}{k^3}$ also converges, and so $\sum_{k=1}^{\infty} \frac{3k^2}{7k^5+2k^2}$ converges by the comparison test.

(b) Divergent. $\lim\limits_{k\to\infty} \frac{k}{\sqrt{k^2+1}} = 1$, so the series diverges by the n-th term test.

(c) Convergent. $\pi^{-n} = \left(\frac{1}{\pi}\right)^n$, so this is a geometric series with ratio $\frac{1}{\pi}$, which is < 1.

(d) Convergent. $\frac{2^n+3^n}{4^n} = \frac{2^n}{4^n} + \frac{3^n}{4^n} = \left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n$, so $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, which is a sum of two convergent geometric series with ratio less than 1. (Or, compare with the geometric series $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^n$.)

(e) Convergent. $\frac{1+\sin(m)}{5^m} \leq \frac{2}{5^m}$, so the series converges by comparison with the convergent geometric series $\sum_{m=1}^{\infty} \frac{2}{5^m}$.

(f) Convergent. $\lim_{n\to\infty} \sqrt[n]{\frac{n^6}{5^n}} = \lim_{n\to\infty} \left(\sqrt[n]{n^5} \cdot \sqrt[n]{\left(\frac{1}{5}\right)^n} \right) = 1 \cdot \frac{1}{5} = \frac{1}{5}$. Since the limit is less than 1, the series converges by the root test. [The ratio test also works.]

(g) Convergent. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left(\frac{((k+1)!)^2}{(k!)^2} \cdot \frac{(2k)!}{(2(k+1))!} \right) = \lim_{k\to\infty} \left((k+1)^2 \cdot \frac{1}{(2k+1)(2k+2)} \right) = \lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2} = \frac{1}{4}$. Since this limit is less than 1, the series converges by the ratio test.

(h) Divergent. $n + \sqrt{n} \leq n + n = 2n$, so $\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{2n}$, and so $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges by the comparison test.

Problem 4. A series of positive and negative terms can either diverge, converge absolutely, or converge conditionally. Classify each of the following series as one of divergent, absolutely convergent, or conditionally convergent. In all cases, explain your reasoning, being explicit about any convergence tests that you apply.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^n}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$

c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ **d)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^k}$

Answer:

(a) Converges absolutely. The series $\sum_{n=0}^{\infty} \left| \frac{(-1)^{n+1}}{3^n} \right| = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series, with ratio $\frac{1}{3}$.

(b) Converges conditionally. The series of absolute values $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ diverges by the comparison test because $\frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k+k}} = \frac{1}{\sqrt{2}} \frac{1}{k^{1/2}}$, and $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ is a divergent p-series. But $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$ converges by the Alternating Series Test, because the sequence $\left\{\frac{1}{k^{1/2}}\right\}_{k=1}^{\infty}$ is decreasing and $\lim_{k\to\infty} \frac{1}{k^{1/2}} = 0$.

(c) Converges conditionally. The series of absolute values diverges by the comparison test because $\frac{\ln(n)}{n} \geq \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges. But $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the Alternating Series Test because $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ and the sequence $\left\{\frac{\ln(n)}{n}\right\}_{n=2}^{\infty}$ is decreasing, at least for $n > 3$. (To see that it is decreasing note that if $f(x) = \frac{ln(x)}{x}$, then $f'(x) = \frac{1-ln(x)}{x^2}$, which is positive for $x > e$.

(d) Converges absolutely. The series of absolute values converges by the root test because $\lim_{k\to\infty} \sqrt[k]{\frac{1}{k^k}} = \lim_{k\to\infty} \frac{1}{k} = 0$.

Problem 5. The series $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ diverges, but none of the tests that we have covered can prove it. Note that $\lim_{n\to\infty} \left(\int_2^n \frac{1}{x\ln(x)} dx \right) = \lim_{n\to\infty} \left(\ln(\ln(x)) - \ln(\ln(2)) \right) = +\infty$. Also note that $f(x) = \frac{1}{x\ln(x)}$ is decreasing. [You do not have to prove these facts.] Show that the partial sum, $s_n = \sum_{k=2}^{n} \frac{1}{k\ln(k)}$, satisfies

$$s_n \geq \int_2^{n+1} \frac{1}{x\ln(x)}\, dx$$

by considering the upper sum using the partition $\{2, 3, 4, \ldots, n+1\}$ of the interval $[2, n+1]$, and conclude that $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ diverges. (Note that this example is a special case of something called the "integral test.")

Answer:

Let $f(x) = \frac{1}{x\ln(x)}$, and let P be the partition of $[2, n+1]$ given by $P = \{2, 3, \ldots, n+1\}$. Note that this partition contains n points and has $n-1$ subintervals.

We know that $U(P, f) \geq \int_2^{n+1} f(x)dx$. Since f is decreasing, we know that the maximum, M_i, of $f(x)$ for x in the i-th subinterval of the partition occurs at the left endpoint, so

$$U(P, f) = \sum_{i=1}^{n-1} M_i(x_i - x_{i-1}) = \sum_{i=2}^{n} f(i) \cdot 1 = \sum_{i=2}^{n} \frac{1}{i\ln(i)}$$

But that means that $U(P, f)$ is precisely the n-th partial sum, s_n of the series $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$. Since $s_n = U(P, f) \geq \int_2^{n+1} \frac{1}{x\ln(x)} dx$ and $\lim_{n\to\infty} \left(\int_2^n \frac{1}{x\ln(x)} dx \right) = +\infty$, we see that the sequence of partial sums is not bounded. So the series diverges.