This test is due by the start of class on the day of the second in-class test, Friday, November 18. **You should do any six out of the seven problems on the test. Do not do all seven; if you do all seven, your answer for problem 7 will be ignored.**

You should do this test on your own, using only your textbook, class notes, and previous work in the course as reference. You should not work with other students, you should not use the Internet or other references, and you should not consult anyone except the professor for the course. You can ask questions about the test in class and by email. I might give some hints and clarifications, but I am unlikely to give extensive, detailed help. Be sure to show all of your work! If you are not able to complete a problem, you should turn in work showing whatever progress you have made on it, for partial credit. **Note: There will be no "rewrites" for this test.**

You can submit your answers in LaTeX on overleaf.com, if you want to do that, but you are welcome to write up your answers neatly by hand.

Although some problems are more difficult than others, and some are fairly easy, all seven problems will count equally.

Problem 1. **(a)** Suppose that the function $F(x)$ is differentiable at a. Show directly from the definition of derivative that the function $G(x) = F(x)^2$ is differentiable at a and $G'(a) = 2F(a)F'(a)$. [Hint: You only need to factor $F(x)^2 - F(a)^2$ in the definition.]

(b) We know that $f(x)g(x) = \frac{1}{4}\big((f(x)+g(x))^2 - (f(x)-g(x))^2\big)$ from a previous homework problem . Using only this fact, the result from part (a), and the sum, difference, and constant multiple rules for derivatives, find the formula for the derivative of $f(x)g(x)$,

Problem 2. Let f and g be differentiable functions on $[a, b]$. Suppose that $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x \in (a, b)$. Prove that $f(b) > g(b)$. [Hint: Consider the function $h(x) = f(x) - g(x)$ and apply the Mean Value Theorem.]

Problem 3. Let f be an integrable function on $[a, b]$. Suppose that $A \le f(x) \le B$ for all $x \in [a, b]$. Show, from the definition of the integral, that $A \cdot (b - a) \le \int_a^b f \le B \cdot (b - a)$. (Hint: Use the trivial partition $P = \{x_0, x_1\}$ where $x_0 = a$, $x_1 = b$.)

Problem 4. Suppose that f is integrable on $[a, b]$. Define $F(x) = \int_a^x f$ for $x \in [a, b]$, and define $G(x) = \int_a^x F$ for $x \in [a, b]$. How do we know $\int_a^x F$ exists? Show that G is differentiable on $[a, b]$.

Problem 5. Let $\sum_{k=1}^{\infty} a_k$ be a convergent series of non-negative terms. Prove that the series $\sum_{k=1}^{\infty} a_k^2$ also converges. [Hints: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and remember that you only need to show $\sum_{k=N}^{\infty} a_k^2$ converges for some N.]

Problem 6 (*Textbook problem 4.5.7, 8*)**.** **(a)** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on an interval I. Assume that each f_n is bounded; that is, there are constants M_n such that $|f_n(x)| \le M_n$ for all $x \in I$. Prove: If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then f must also be bounded on I.

(b) Show that the hypothesis of uniform convergence is necessary by finding a sequence of bounded functions that converges pointwise to a function that is not bounded. ([Hint: Take $I = [0, \infty)$ and look for a simple example.]

Problem 7. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le r|x - y|$ for all $x, y \in \mathbb{R}$, where r is a constant in the interval $0 \le r < 1$. Such a function is said to be a **contraction** on $\mathbb{R}$. Note that a contraction is simply a Lipschitz function with Lipschitz constant strictly less than 1, so we already know that f is continuous.

(a) Let t be any real number. Define a sequence $\{a_n\}_{n=0}^{\infty}$ by $a_0 = t$, $a_n = f(a_{n-1})$ for $n > 0$. That is $a_0 = t, a_1 = f(t), a_2 = f(f(t)), a_3 = f(f(f(t))), \dots, a_n = f^n(t), \dots$, where f^n is the composition of f with itself n times. Show that the sequence $\{a_n\}_{n=0}^{\infty}$ is contracting, and hence is convergent.

(b) Let $z = \lim_{n\to\infty} a_n$. Show that $f(z) = z$, that is, z is a fixed point of f. [Hint: Write $f(z) = f\big(\lim_{n\to\infty} a_n\big) = f\big(\lim_{n\to\infty} f^n(t)\big)$. and use the fact that f is continuous.]

(Note: Recall that a **fixed point** of a function f is a point y such that $f(y) = y$. It is clear that a contraction can have at most one fixed point. This problem shows that a contraction always does have a fixed point. Furthermore, if t is any real number, then the sequence $\{f^n(t)\}_{n=0}^{\infty}$ converges to that unique fixed point. This is the **Contraction Mapping Theorem** for $\mathbb{R}$.)